Towards an operad of goals

Steve Huntsman

BAE Systems FAST Labs (goo.gl/Vrsqgm)

15 March 2018
Disclaimer

This is deliberately preliminary and hopefully provocative
Autonomous systems of systems (SoSs)

- Autonomous SoSs require us to design properties that are
  - Global (⇒ topology)
  - Emergent (⇒ statistical physics)
  - Compositional (⇒ category theory)

- Very rare to see work that applies even two of these fields

- Goal: sketch one way that all three might usefully contribute
  - Simplicial sets (roles), sheaves (data fusion), . . . skipping this
  - Renormalization
  - Operads

- Ideal: model SoSs with a compositional framework capable of capturing emergent and global properties
  - Account for variation through scoped parameters/settings
Statistical physics: renormalization “group”

- Physical perspective informed by Kadanoff, Wilson, *et al.*
  - Theory given by a function $f(\xi; \pi)$ of data $\xi$ and parameters $\pi$
  - If for a suitable coarsening operator $C$ there exists $g$ s.t.
    $f(\xi; \pi) = f(C(\xi); g(\pi))$ then theory is renormalizable
  - Fixed point for infinite domains is of great physical interest
- The nLab features a very fancy QFT-oriented treatment that goes in the direction of refinement vs. coarsening
  - Higher-order Feynman diagrams (insertion operad!)
- **We are not going to do anything fancy in what’s next**
  - $f$ is a weighted average of finitely many function translates
  - $\xi$ is the set of translation points; $\pi$ is the weighting
  - $C$ clusters translation points (BTW, single linkage is a functor)
  - $g$ renormalizes a probability distribution based on clustering
The monopole operad

- Intuition: in far field, many point masses act like a single one
- Ingredients
  - Little disks operad
  - Probability operad
  - “Homogeneous” potential \( V(r) := r^{-k} \) or \(-\log r\)
- Sources and weights \((\xi, \pi) \in (\mathbb{R}^d)^n \times \Delta^n\)
- Operations: \( V_{\xi,\pi}(x) := \sum_j \pi_j V(|x - \xi_j|) \sim V(|x|) \)
- Composition also \( \sim V(|x|) \):

\[
V_{\xi,\pi} \circ \left( V_{\xi^{(1)},\pi^{(1)}}^{(1)}, \cdots, V_{\xi^{(n)},\pi^{(n)}}^{(n)} \right)(x) := \sum_j \pi_j V_{\xi^{(j)},\pi^{(j)}}(x - \xi) \]
So what?

- Ideal: model SoSs with a compositional framework suitable for engineering emergent and global properties
  - Account for details with scoped parameters, decorations, etc.
- Example: want a multiagent system that features
  - Distributed planning with $\ll n^2$ communications overhead
  - Robustness under addition/subtraction of agents
- Possible ingredients for (de)composing:
  - Tasks (flow graph operad)
    - Slides at math.ucr.edu/home/baez/ACT2017
    - Some notion of geometry necessary for motion planning
  - Constraints ("SAT operad")
  - Teams (simplicial operad)
    - Topology characterizes horizontal relationships
  - Communications (want an operadic framework!)
Artificial potentials

- Classic alternative to discretizing space for motion planning
  - Involves attractive (goal) and repulsive (obstacle) terms
    - Agents repel each other
    - Troublesome metastable minima can occur
    - Alas, the monopole operad can’t handle signed terms
- Total potential must represent complex spatial relationships
  - Many terms = expensive
- Might still be useful for applications where space is discrete
  - E.g., graph Laplacian stuff
  - Not pulling on this thread here
The fast multipole method (FMM)

- FMM [Greengard & Rokhlin 1987] accurately simulates \( N \) interacting particles in \( O(N) \) (vs \( O(N^2) \)) time
  - Named one of the top 10 algorithms of the 20th century [Dongarra & Sullivan (2000)]
- Two key ideas (approximate renormalization!)
  - Represent clusters of particles with multipole expansions that are as coarse (i.e., truncated) as possible in far-field
    - For approximation error \( \varepsilon \), truncate at \( \lceil \log_2(1/\varepsilon) \rceil \) terms
    - Nearby particles interact directly
  - Decompose space hierarchically to get well-separated clusters
- Works for (at least) homogeneous and logarithmic potentials
- Parallel version [Greengard & Gropp 1990] exploits locality

Approved for public release; distribution unlimited
The fast multipole method (FMM)

Eval points $\sim \frac{4}{5} U(\text{top half}) + \frac{1}{5} U(\text{bottom half})$

Red: $+$ source; blue: $-$ source; black: gradient evals
A multipole operad?

- The FMM builds a tree whose nodes are decorated with truncated multipoles
  - Approximates finer quadtree with monopoles on leaves
  - Tree topology doesn’t depend on charge values
- We’ll show how to use the FMM to build networks that seem to play nicely with
  - Little cubes operad
  - Motion plans
- This points towards our desiderata, viz.
  - Distributed planning with $\ll n^2$ communications overhead
  - Robustness under addition/subtraction of agents
- Exercise: how much of this can be realized operadically?
Coordination topology

- Connect all nodes in the same leaf box
- Connect nearest nodes in adjacent boxes
- Connect orphans to nearest nodes (not shown)
Coordination topology

- Connect all nodes in the same leaf box
- Connect nearest nodes in adjacent boxes
- Connect orphans to nearest nodes (not shown)
Comparison with random geometric graphs

\[ \text{efficiency} = \langle d_{jk}^{-1} \rangle_{j \neq k} \]
Comparison with random geometric graphs

\[
\text{efficiency} = \langle d_{jk}^{-1} \rangle_{j \neq k}
\]
Comparison with random geometric graphs

\[ \text{efficiency} = \langle d_{jk}^{-1} \rangle_{j \neq k} \]
Towards an operad of goals

- FMM-based framework for networks and motion plans
  - How much of this can be realized operadically?
- Flow graph operad for tasks
- SAT operad for constraints
  - Weighted MAX-SAT for optimization
- Simplicial operad for teams and requirements
  - Matching a team to requirements amounts to finding an optimal (not necessarily surjective) simplicial map
- Topology, statistical physics, and category theory can (and likely will) have something to contribute to the development of autonomous systems of systems
Thanks

BAE Systems FAST Labs (goo.gl/Vrsqgm)

steve.huntsman@baesystems.com
Moving parts for classical case

- \( \text{dim} = 2, \ V = -\log, \) charges \( q_j \) at \( z_j \in \mathbb{C}; \) take real parts

- Set \( Q := \sum_j q_j; \ A := \sum_j |q_j|; \ a_\ell := - \sum_j q_j z_j^\ell / \ell; \ r > \max |z_j| \)

- Multipole expansion with error bounds:

\[
\left| \sum_j q_j \log(z - z_j) - Q \log z - \sum_{\ell=1}^{L} \frac{a_\ell}{z^\ell} \right|_{|z|>r} \leq \frac{A}{\left( \frac{|z|}{r} - 1 \right) \left( \frac{|z|}{r} \right)^L}
\]

- More involved error bounds for shifting centers of multipole expansions, converting them to Taylor series, and shifting the Taylor expansions
Topology: simplicial sets, sheaves, etc.

Subsystem interfaces $I_j$
Sensors $S_j$
Effectors $E_j$
Topology: simplicial sets, sheaves, etc.

$k$-simplices: roles with $k - 1$ players
Subsystem interfaces $I_j$
Sensors $S_j$
Effectors $E_j$
Fusing sensors $\Rightarrow$ nontrivial homology