A Category Theoretical Investigation of the Type Hierarchy for Heterogeneous Sensor Integration

CLIFF JOSLYN, EMILIE PURVINE, MICHAEL ROBINSON

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Outline

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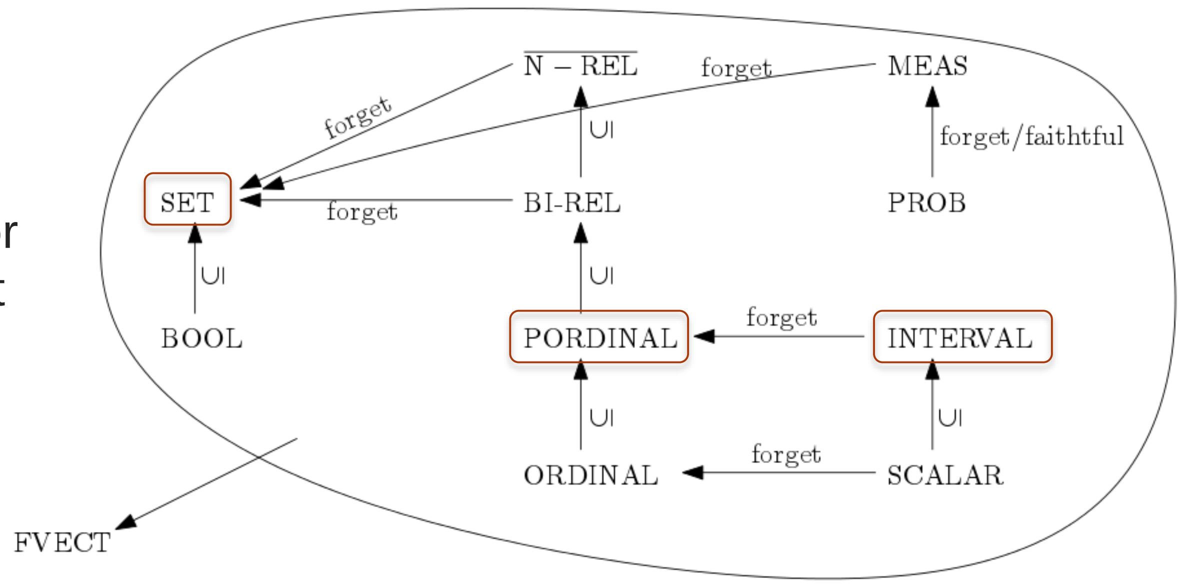
- Setting the stage – information integration scenario [Michael Robinson's flash talk]
- Our strategy to encode data into a vector space
- Examples
  - SET
  - PORDINAL
  - INTERVAL – a work in progress

Data type hierarchy



- Data types fall into a few categories – literally!
  - Can create categories for each type of data so that stalks are objects in that category
- Transform between data types using *functors*
  - Lose some information e.g., SCALAR → ORDINAL we lose addition
- Transformation to FVECT allows us to turn any data type into a finite dimensional vector space, keeping structure

Emilie Purvine, Cliff Joslyn, Michael Robinson. *A Category Theoretical Investigation of the Type Hierarchy for Heterogeneous Sensor Integration*. https://arxiv.org/abs/1609.02883

Our strategy

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- Given a single stalk, or data type, e.g., set, boolean, partial ordinal, ordinal, interval, scalar, probability, binary relation…
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- Bonus: if we put structure of c into D via morphisms then we can get a map $V \to V$ which encodes this structure – the adjacency matrix of morphisms in Decat(D)

Example – categorical data

c = {police officer, protester, bystander}

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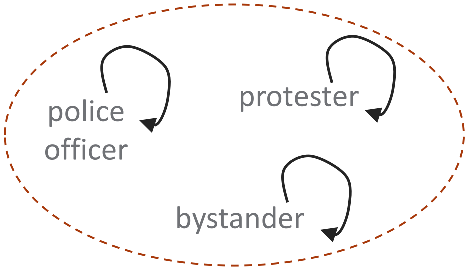

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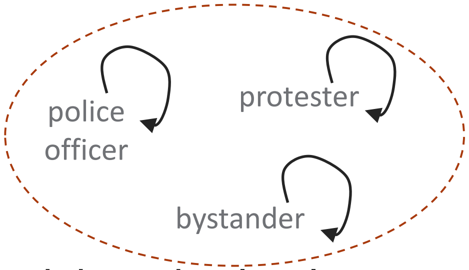

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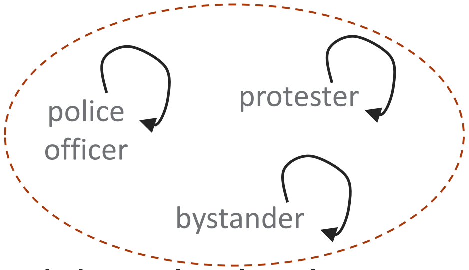

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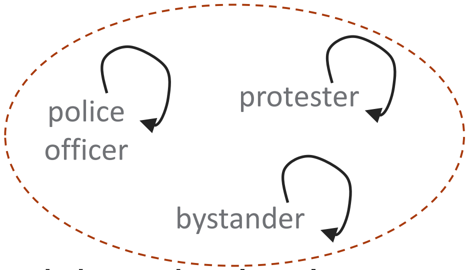
$D =$ 

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- Bonus: There is no structure on c (or any categorical) which is seen by the absence of morphisms other than identities in D. Map $V \rightarrow V$ is just the identity matrix in this case

Example – categorical data (cont.)

- Replace stalk {police officer, protester, bystander} with new stalk $\mathbb{R}$[{ {police officer}, {protester}, {bystander} }]
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  - …
- Interpretation must be situation dependent

Example – partial order data

c = [{cat, dog, mammal}, {cat ≤ mammal, dog ≤ mammal}]

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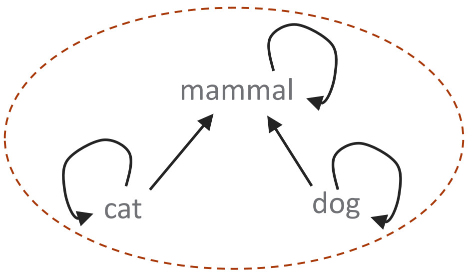

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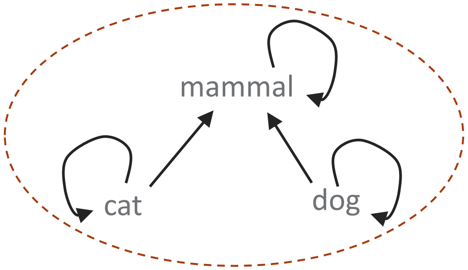
$D =$ 

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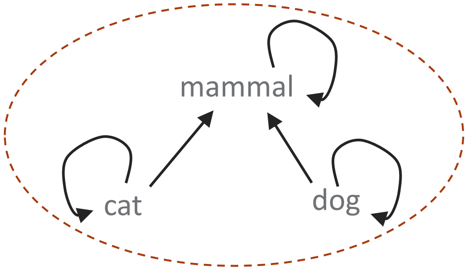
$D =$ 

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- Bonus: There is structure on c (and all partial orders) which is seen through the morphisms in D encoding the partial order relation.
  - Basis elements mapped to their principle filter

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

Example in progress – real interval valued data

$c = \{[a, b] : a, b \in \mathbb{R}, a \leq b\}$ with structure of a semiring (addition, multiplication, identities are defined, but not inverses) and a partial order relation $[a, b] \leq [c, d]$ if and only if $a \leq c, b \leq d$

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- In this case, when *c* already has a lot of structure, going through an intermediate category *D* is probably more trouble than is needed
  - A ring can be encoded as a preadditive category – a category in which all Hom sets have the structure of an abelian group – with one object
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- Instead, starting with fields – closest to vector spaces – and working down to semirings
  - For field *F* “categorify” as vector space F^1
  - Work in progress to “categorify” rings and then semirings

Summary & Future Work

- Real-world data integration tasks need a way to put different data types on the same footing
- Finite dimensional vector spaces give that common footing and allow topological invariants to be calculated
- We developed a three step process and showed examples for three different data types
- Future work:
  - Implementing this into existing sheaf code packages
  - Showing examples using real data
  - Exploring how distances in the original sets get transformed