A higher-order temporal logic for dynamical systems

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An example system

The National Airspace System (NAS).

- Goals of NextGen:
  - Double the number of airplanes in the sky;
  - Remain extremely safe.

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- Systems of systems:
  - A great variety of interconnected systems.
  - Work in concert to enforce global property: safe separation.

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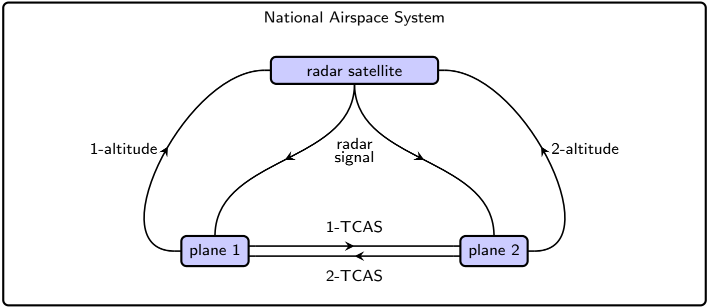
Systems of interacting systems in the NAS



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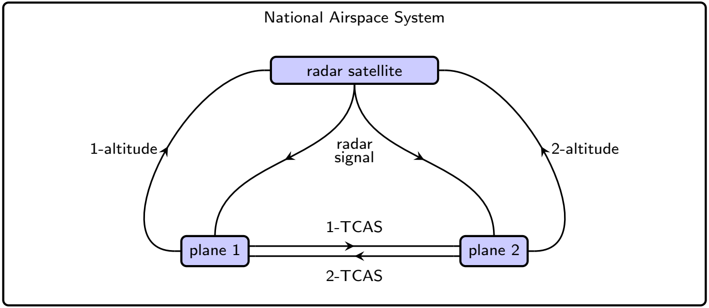
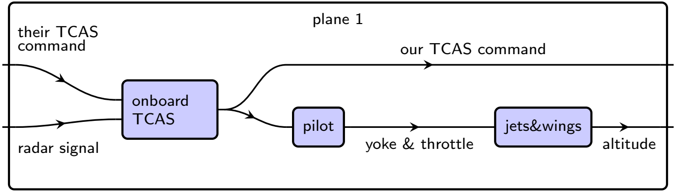
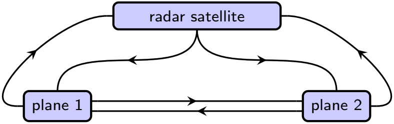




A hypergraph category



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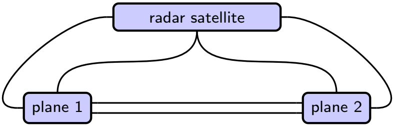

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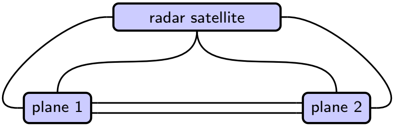

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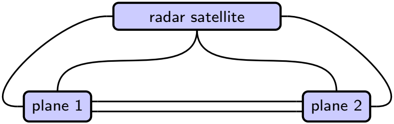
- What are these pictures formally?
  - Composition diagrams in a *hypergraph category*.
  - What we called "constraints" are formalized as relations.

Relations in a topos



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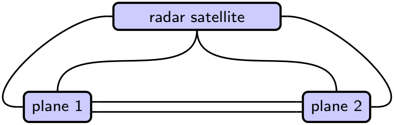

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- Example: relations in $\mathcal{E} = \mathbf{Set}$.
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- Relations on $A = A_1 \times \cdots \times A_n$ are morphisms $A \to \Omega$.

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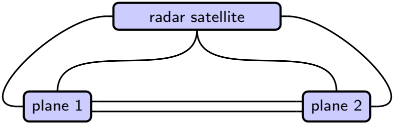

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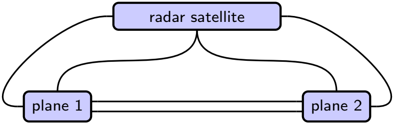

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- More generally, where do all these behaviors live?
- They live in time.
- Goal: a good topos for studying behaviors (hence time).

NAS use-case as guide

What's the topos for the National Airspace System?

- This question was a major guide for our work.
- Need to combine many common frameworks into a "big tent".
  - Differential equations, continuous dynamical systems.
  - Labeled transition systems, discrete dynamical systems.
  - Delays, non-instantaneous rules.
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Relationship to toposes:

- Toposes have an associated internal language and logic.
- Can use formal methods (proof assistants) to prove properties of NAS.

Plan of the talk

1. Define a topos $\mathcal{B}$ of behavior types.
2. Briefly discuss *temporal type theory*, which is sound in $\mathcal{B}$.
3. Return to our NAS use-case.

What is a topos and why?

Toposes—invented by Grothendieck—generalize topological spaces.

- Basic idea:
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Answer: roughly, Time. But what is that?

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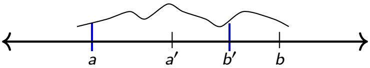

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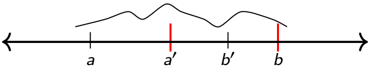

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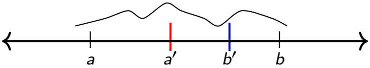

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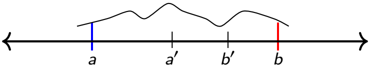

Why $\mathbb{R}$ is not preferable as the site

Two reasons *not to use* $\mathrm{Shv}(\mathbb{R})$ as our topos.

- 1. Often want to consider **non-composable** behaviors!
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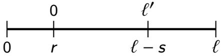
Discard composition gluing, add translation invariance.

Our choice of topos $\mathcal{B}$

Use $\mathbb{IR}_{/\rhd}$ the following site:

- Objects $= \{\ell \in \mathbb{R}_{\geq 0}\}$.
- $\mathrm{Hom}(\ell', \ell) = \{\langle r, s\rangle \mid r + \ell' + s = \ell\}$



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- When $r, s > 0$, write $\ell' \rightsquigarrow \ell$.[2]

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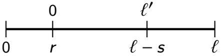

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The topos of behavior types: $\mathcal{B} = \mathrm{Shv}(\mathbb{IR}_{/\triangleright})$.

- A sheaf X assigns a set of possible behaviors to each ℓ,
- And a restriction map to each included subinterval $\langle r, s\rangle \colon \ell' \to \ell$,
- Such that $X(\ell) \cong \lim_{\ell' \rightsquigarrow \ell} X(\ell')$.

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- E.g. simply-typed lambda calculus with sum types and quotient types.
  - Start with atomic types and atomic terms.
  - Build new types, terms, and propositions using constructors.
    - Types: $\mathbb{N}$, $\mathtt{Prop}$, products, arrows, sums, quotients.
    - Terms: tupling, projection, lambda abstraction, evaluation, etc.
    - Propositions: $\exists, \forall, \wedge, \vee, \neg, \Rightarrow, \Leftrightarrow, \top, \bot$.
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I thought this was dreadfully boring. Until I witnessed...

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- Start with atomic types, terms, and axioms from your topos.
- Kripke-Joyal is a machine that turns logic into topos-proofs.
- Suppose you have any expression in the type theory.
  - It automatically has semantics in your topos.
  - That is, it means something about sheaves X.
  - $\forall(x : X)$ – "for all restriction maps and sections x..."
  - $\exists(x : X)$ – "there is a covering family and a section x in each..."
  - Each connective $\wedge, \vee, \Rightarrow$, means something sheafy.
- Statements and proofs are recursive, tree-like structures.
  - Kripke-Joyal recurses over that structure.
  - At each step, it unwinds the logic into restrictions, covers, sections.
  - It manages all the topos stuff and lets you believe you're in **Set**.

The Kripke-Joyal semantics: doing the heavy lifting.

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  - A group object in this topos is a group that can change in time.
  - A database schema is one that can change in time.
- There is a type $\mathbb{R}_{\mathsf{var}}$ of real numbers that change continuously in time.
  - It is a topological ring object just like real numbers always are.
  - Define temporal derivatives, rate of change through time, within the logic.
  - We prove logically that it satisfies the usual rules (linear, Leibniz)
  - And we check semantically that it actually is the derivative.

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$$f(\dot{x}, \ddot{x}, a, b) = 0.$$

- Maybe $a, b : \mathbb{R}_{\mathrm{var}}$ are continuous functions of time.
- Regardless, $f(\dot{x}, \ddot{x}, a, b) = 0$ is just an equation in the logic.
  - Use it with $\top, \bot, \neg, \vee, \wedge, \Rightarrow, \exists, \forall$.
  - Can be combined with any other property.

The problem: safe altitude

Simplifying the safe separation problem.

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Goal: combine disparate guarantees to prove useful result.

Setup

Variables to be used, and their types:

$$t : \mathtt{Time}. \quad T, P : \mathrm{Cmnd}. \quad a : \mathbb{R}_{\pi}. \quad \mathtt{safe}, \mathtt{margin}, \mathtt{del}, \mathtt{rate} : \mathbb{Q}.$$

What these mean:

$t : \mathtt{Time}$.	time-line	(a clock).
$a : \mathbb{R}_{\mathrm{var}}$.	altitude	(continuously changing).
$T : \mathrm{Cmnd}$.	TCAS command	(occurs at discrete instants).
$P : \mathrm{Cmnd}$.	pilot's command	(occurs at discrete instants).
$\mathtt{safe} : \mathbb{Q}$.	safe altitude	(constant).
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Behavior contracts

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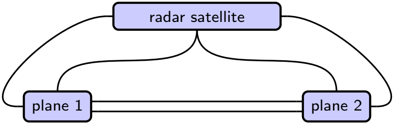
- Can prove safe separation

$$\forall (t : \mathtt{Time}). \downarrow_0^t \left(t > \mathtt{del} + \frac{\mathtt{safe}}{\mathtt{rate}} \;\Rightarrow\; a \geq \mathtt{safe}\right).$$

Summary



- Idea: topos theory for integrating systems in a big tent.
- Many different formalisms for behavior, but they all occur in time.
  - We say that time occurs in intervals, which can be restricted.
  - Sheaves are behavior types: what can occur over intervals.
- The topos has a native "internal" logic.
  - Looks like usual set theory, $\forall, \exists, \wedge, \vee, \Rightarrow, \neg$; use formal methods
  - It compiles via Kripke-Joyal into complex facts about sheaves.

This temporal type theory is quite general, and fully compositional.

If you're interested in reading more

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Questions and comments are welcome. Thanks!