Seeking a Categorical Systems Theory via the Category of Hypergraphs

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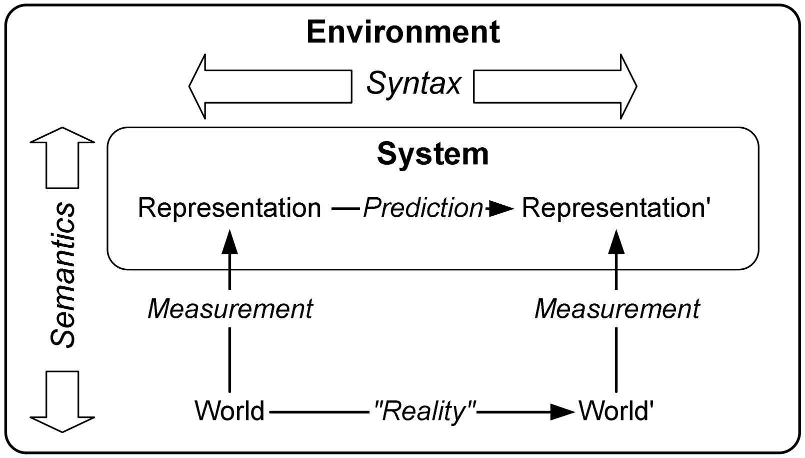
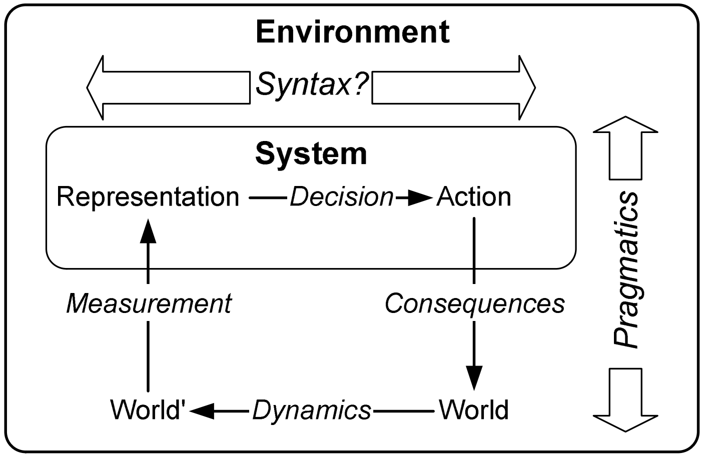
NIST Workshop on Applied Category Theory: Bridging Theory and Practice
March 15-16, 2018
PNNL-SA-133059

Mathematical Systems Theory

- **Intellectual Tradition:** Complex systems, general systems theory, systems science, cybernetics, semiotics
- **Interdisciplinary models of *evolving, hierarchical, semiotic information control systems:***
  - Physics, biology, ecology, neurology, economics, robotics, socio-technical
- **Classes of *formal models:*** Interrelation, translation, integration
- Philosophical affinity with *universal modeling frameworks*
  - **Automata Theory:** Arbib, Eilenberg, Zeigler
  - **Logic and CS formalisms:** UML, FOL
  - **Formal Ontologies:** OWL, Conceptual Graphs (Sowa)
- **Mathematics:** Category theory



Modeling Relation



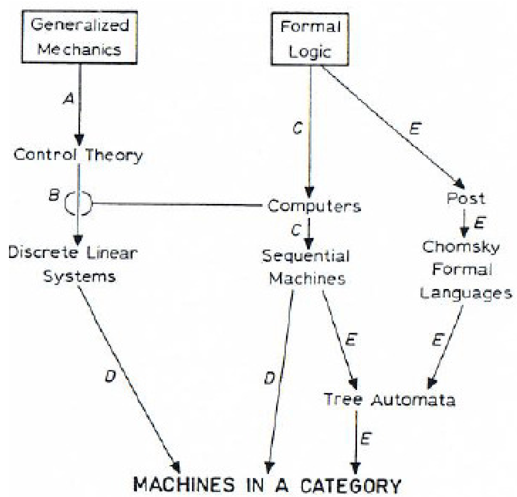
Control Relation

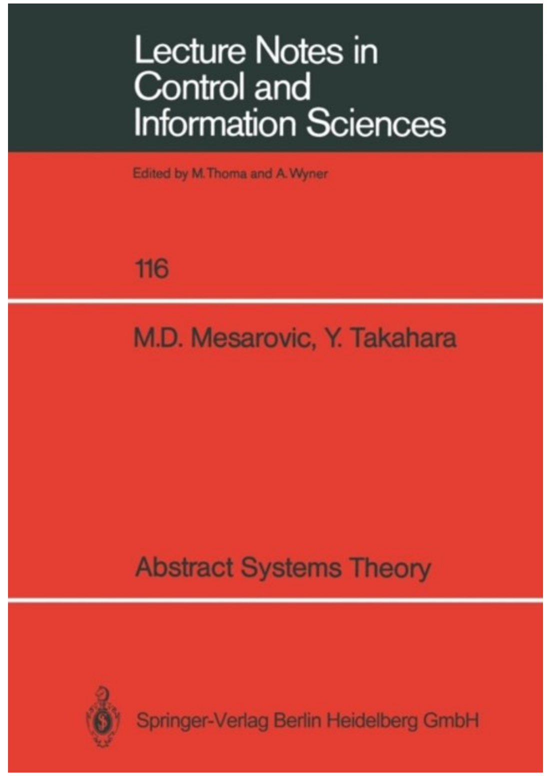
Category Theory Historical in Systems Science

A CATEGORY-THEORETIC APPROACH TO SYSTEMS IN A FUZZY WORLD*

MICHAEL A. ARBIB AND ERNEST G. MANES

(1977) in: Systems: Approaches, Theories, Applications, ed. WE Hartnett, pp. 1-26, D. Reidel

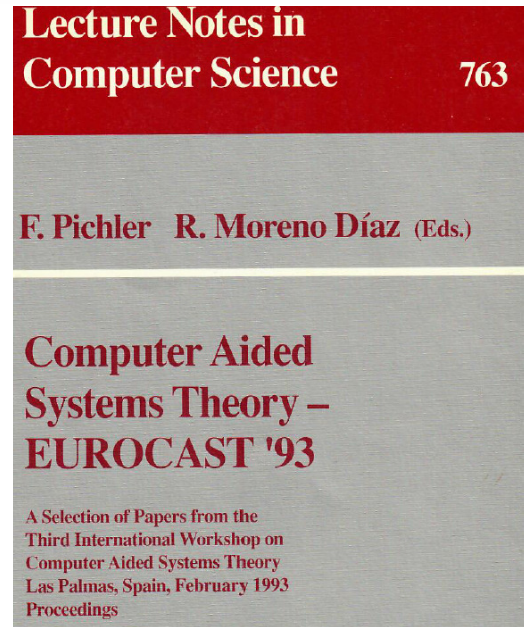




The Shape of Complex Systems

pp. 72-82

Charles Rattray



Mesarovic, MD and Takahara, Y: (1988) ***Abstract Systems Theory*****, Springer-Verlag**

Takahara, Y and Takai, T: (1985) "Category Theoretical Framework of General Systems", Int. J. General Systems, 11:1, pp. 1-33

Resconi, Germano and Jessel, Maurice: (1986) "General System Logical Theory", Int. J. General Systems, v. 12, pp. 159-182

Approaches to Categorical Semiotics

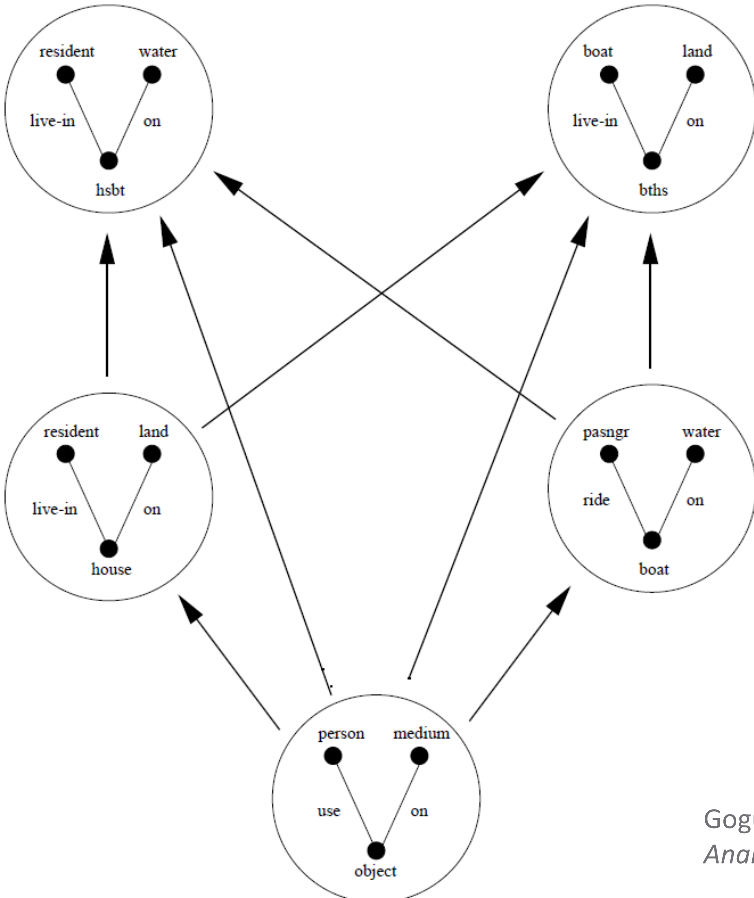


Figure 6: Two Different Blends of Two Input Spaces

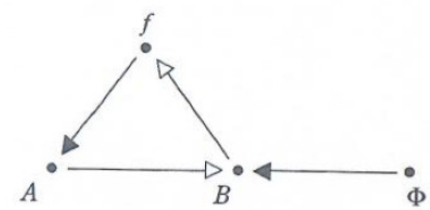
Goguen, JA: (1999) "An Introduction to Algebraic Semiotics, with Application to User Interface Design", in: *Computation for Metaphor, Analogy and Agents, LNAI*, 1562, ed. Chrystopher Nehaniv, pp. 242-291

L I F E I T S E L F

A Comprehensive Inquiry Into the Nature, Origin, and Fabrication of Life

Rosen, Robert: (1991) *Life Itself*, Columbia U Press, New York

250 *Life Itself: The Preliminary Steps*



[10C.3]

which we can correspondingly abbreviate as

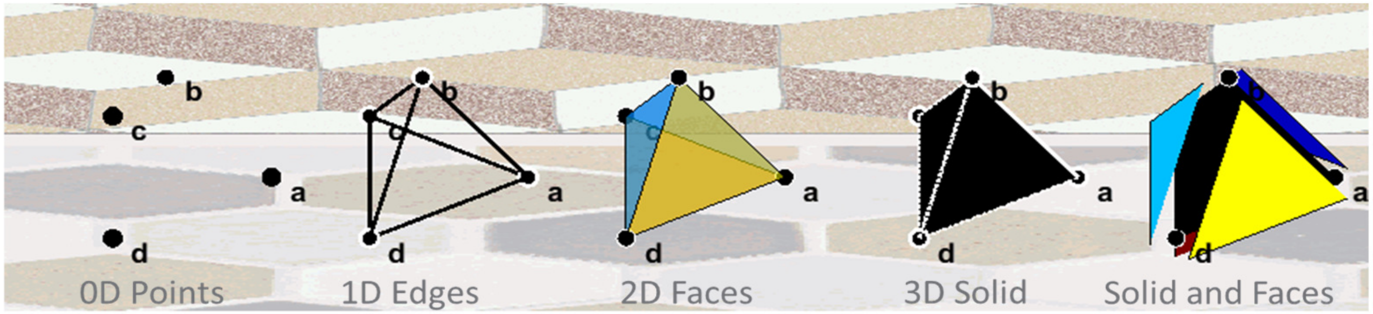
$$A \xrightarrow{f} B \xrightarrow{\Phi} H(A, B). \qquad [10C.4]$$

Goguen, JA: (1967) "L-fuzzy Sets", *J. Math. Analysis and Appl.*, 18, pp. 145-174

Goguen, JA: (1991) "A Categorical Manifesto", Mathematical Structures in Computer Science, 1:1, pp. 49-67

Goguen, JA: (1992) "Sheaf Semantics for Concurrent Interacting Objects", *Math. Structures in Computer Science*, 2:2, pp. 145-174

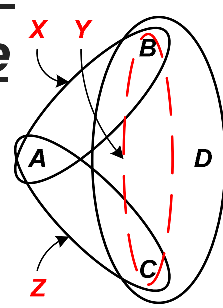
Years Later: Complex Systems Modeling With Finite Set Systems

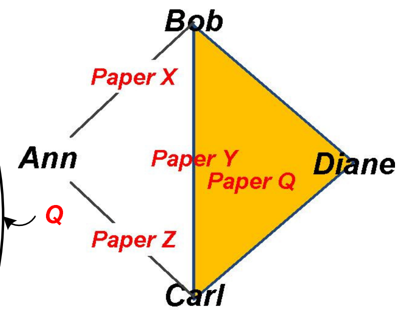


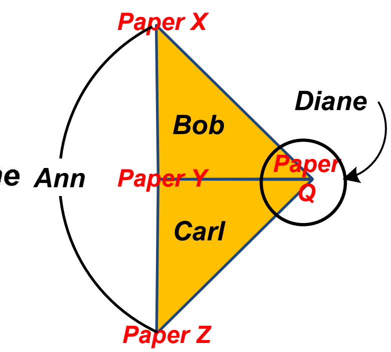
$\{a,b,c,d\}$ $\{ab,ac,ad,bc,bd,cd\}$ $\{abc,abd,acd,bcd\}$ $\{abcd\}$ $\{abcd,abc,abd,acd,bcd\}$

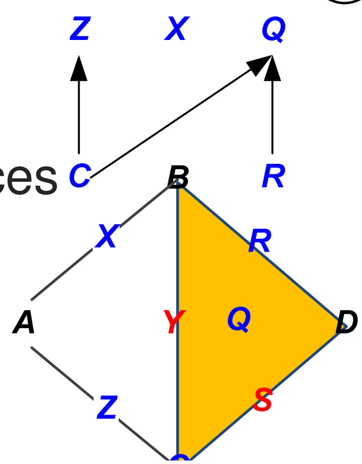
	X	Y	Z	Q
A	x		x	
B	x	x		x
C		x	x	x
D				x

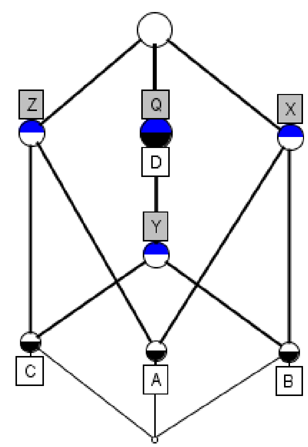
- ***Base spaces of simplicial sheaves***
- ***Multidimensional network science***
- (Transposable) incidence matrices
- (Dual) hypergraphs
  - Abstract simplicial complexes
  - Sperner families
- Finite topologies
- (Dual) finite orders and lattices
- Bipartite graphs
- (Dual) concept lattices

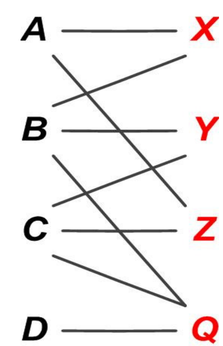




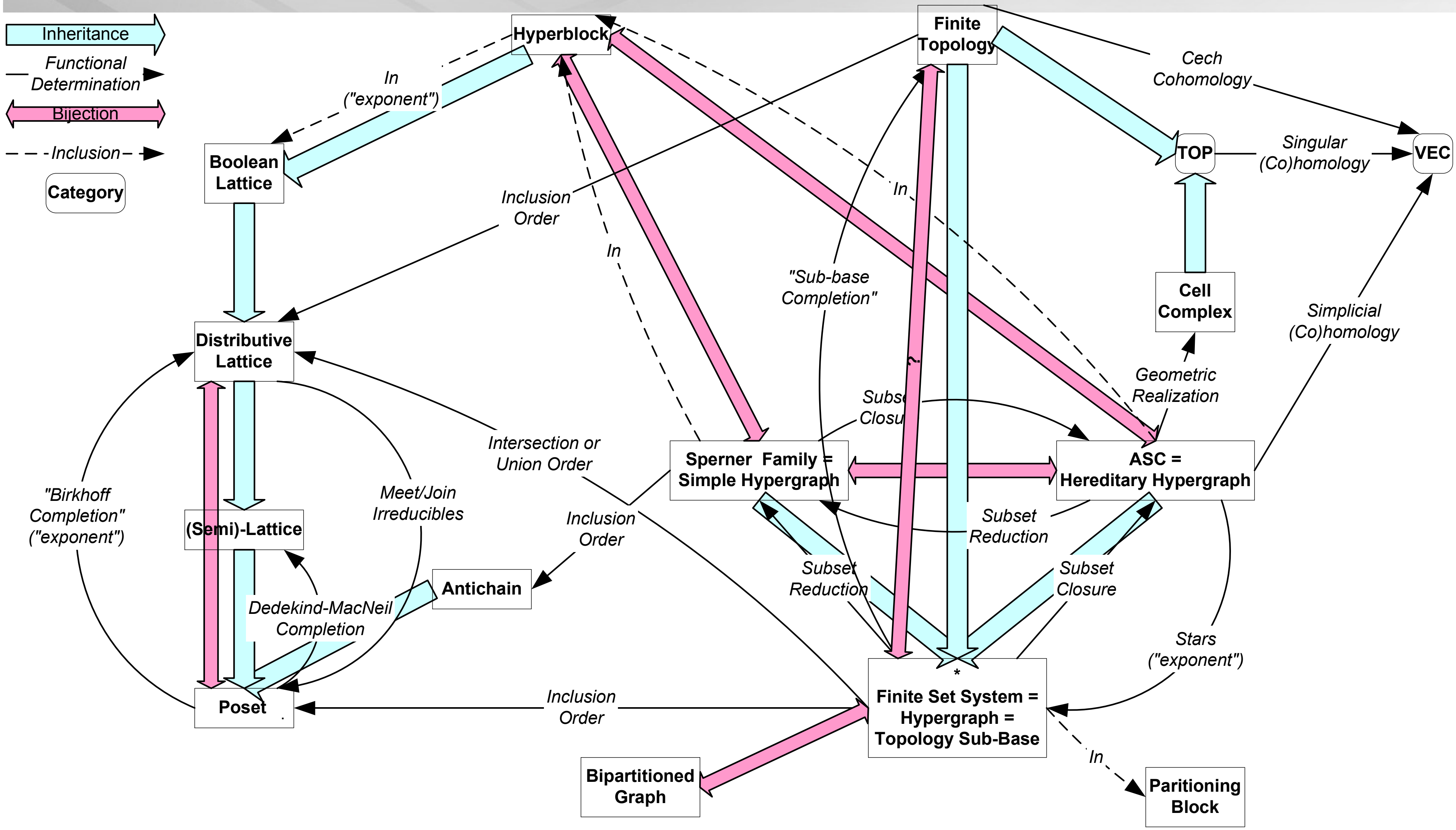




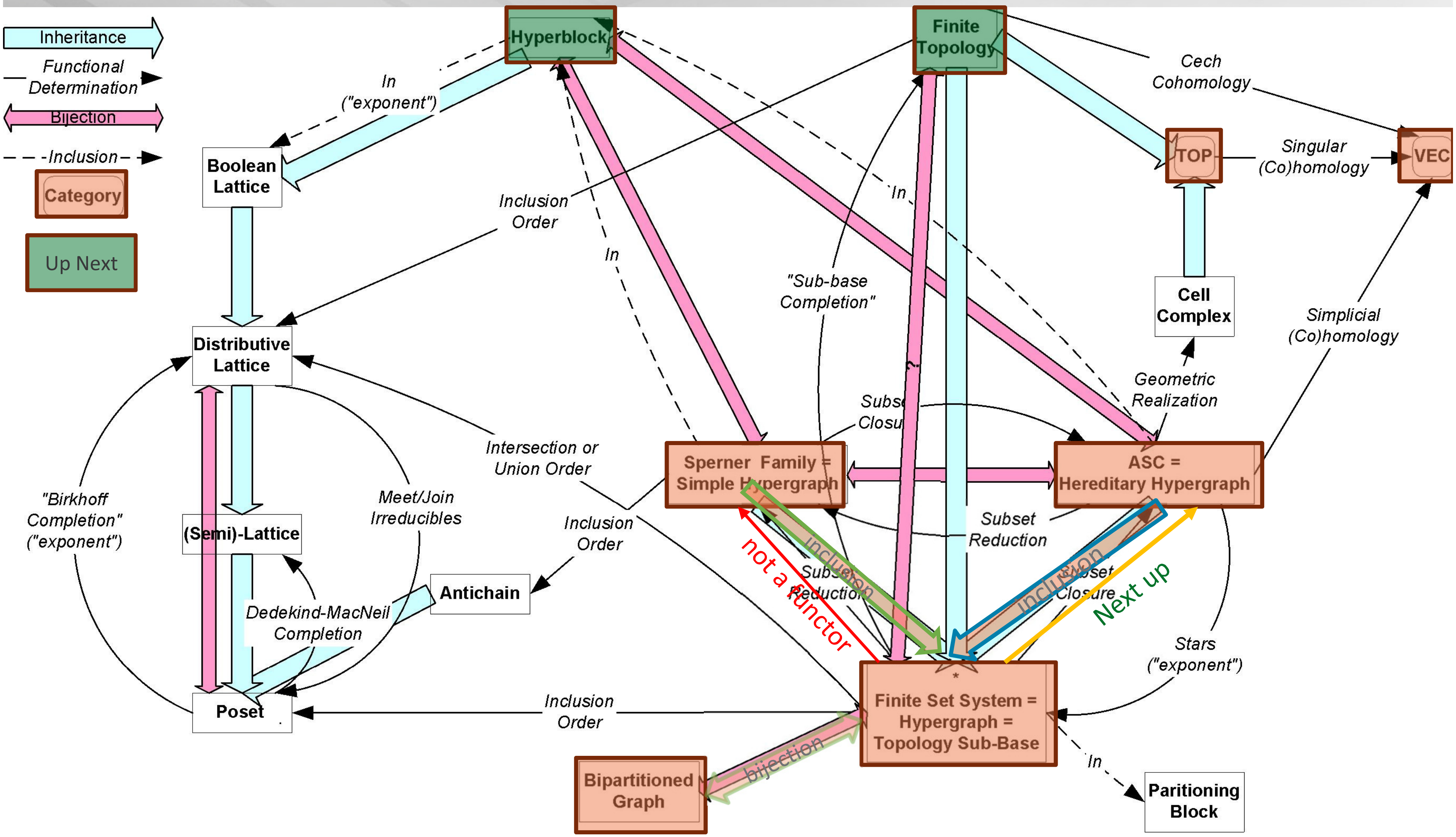




Our Current Map of the World



And It's (Partially Pending) "Categorialization"



Concrete Question

- **Hypergraphs for Community Interaction Analytics**
  - **Collaboration hypernetworks:** Email, proteomics, cyber, neurology
- **Question from our sponsor:** *Why hypergraphs?*
  - Why not just do bipartite graph analysis?
  - (How) Are the categories of finite hypergraphs, bipartite graphs, and Boolean matrices "equivalent"?
- **Answer:** It depends on definitions and cases (of course):
  - Isolated vertices
  - Self-loops
  - Multi-edges
  - Empty hyperedges
  - Empty hypergraphs
- **And apparently, especially:** *Preservation of duality*
  - Subgraph isomorphism is the stock in trade graph analytics
  - Shouldn't sub-hypergraph isomorphism then also be for hypergraph analytics?
  - **But:** Then duality is violated!

Category of Hypergraphs

Objects: Let $\mathcal{H} = (V, E, f)$ where V is a set of vertices, E is a set of edges, and $f : E \to 2^V$ maps each edge to its set of vertices. We require that $V \cap E = \emptyset$. We call such an $\mathcal{H}$ a *hypergraph*.

Arrows: Let $\mathcal{H}_1 = (V_1, E_1, f_1)$ and $\mathcal{H}_2 = (V_2, E_2, f_2)$ be two hypergraphs. A *hypergraph homomorphism* is a pair of set morphisms, $h_V : V_1 \to V_2$, $h_E : E_1 \to E_2$, such that the following diagram commutes:

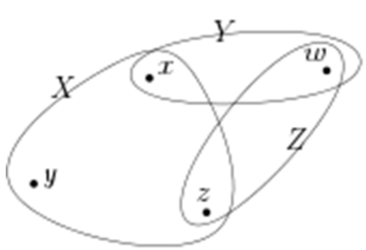
$$\begin{tikzcd} E_1 \arrow[r, "f_1"] \arrow[d, "h_E"'] & 2^{V_1} \arrow[d, "P(h_V)"] \\ E_2 \arrow[r, "f_2"'] & 2^{V_2} \end{tikzcd}$$

Dörfler, W., and D. A. Waller. "A category-theoretical approach to hypergraphs." *Archiv der Mathematik* 34.1 (1980): 185-192.

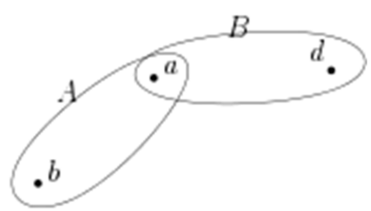
Idea: A vertex map that carries the edges around with it.

- Inclusion map (subhypergraph)
- Isomorphisms
- Collapsing vertices and edges

Example:



$\mathcal{H}_1$



$\mathcal{H}_2$

$$h_V : \begin{cases} x \mapsto a \\ y \mapsto b \\ z \mapsto a \\ w \mapsto d \end{cases} \qquad h_E : \begin{cases} X \mapsto A \\ Y \mapsto B \\ Z \mapsto B \end{cases}$$

Subcategories and Duality

- GRAPH is a subcategory of HG
- BIP (= Bipartite graphs) is a subcategory of GRAPH
- Desire:
  - Duality of hypergraph is a functor from HG to itself

Let $d : \mathrm{HG} \to \mathrm{HG}$ be defined as follows:

$$d(\mathcal{H}) = \mathcal{H}^* = (E, V, f^*)$$

$$d(\mathcal{H}_1 \xrightarrow{(h_V, h_E)} \mathcal{H}_2) = d((h_V, h_E)) : d(\mathcal{H}_1) \to d(\mathcal{H}_2)$$

where $f^* : V \to 2^E$, $f^*(v) = \{e \in E : v \in f(e)\}$, and $d((h_V, h_E)) = (h_E, h_V)$

$$\begin{tikzcd} \mathrm{HG} \\ \textsc{Graph} \arrow[u, hook] \\ \textsc{Bip} \arrow[u, hook] \end{tikzcd}$$

- Problem: *$d(h_V, h_E)$* is not always a morphism
  - Example: $\mathcal{H}_1 \subset \mathcal{H}_2$ and h_V, h_E are inclusion
- Solution: Require morphisms to be *surjective*

Surjective Hypergraph categories

- Require all morphisms to be surjective in HG, GRAPH, BIP

$$
\begin{tikzcd}
\mathrm{HG} & \underline{\mathrm{HG}} \arrow[l, hook'] \arrow[loop above, "duality"] \\
\textsc{Graph} \arrow[u, hook] & \underline{\textsc{Graph}} \arrow[l, hook'] \arrow[u, hook] \\
\textsc{Bip} \arrow[u, hook] & \underline{\textsc{Bip}} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}
$$

Bicolored graphs

- Bicolored graph = Bipartite graph with a 2-coloring specified

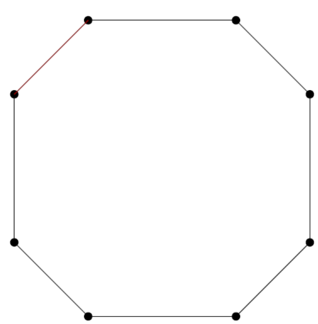
Objects: A *bicolored graph* is a triple (V, E, c), where c is a bicoloring of $G = (V, E)$.

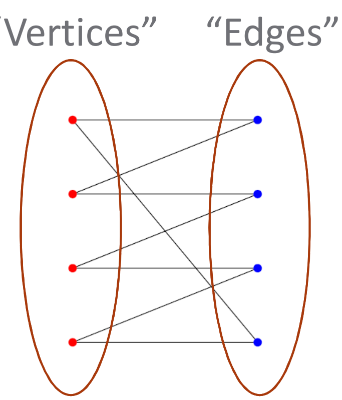
Arrows: Given $G_1 = (V_1, E_1, c_1)$ and $G_2 = (V_2, E_2, c_2)$, two bicolored graphs, a *surjective) bicolored graph homomorphism* is a (surjective) graph homomorphism, $h : V_1 \to V_2$, such that

$$c_1^{-1}(0) = h^{-1}(c_2^{-1}(0)), \qquad c_1^{-1}(1) = h^{-1}(c_2^{-1}(1)).$$

In other words, h must map color 0 vertices to color 0 vertices, and color 1 vertices to color 1 vertices.

Bipartite





Bicolored

Full map of Hyper- and Bicolored Graphs

$$
\begin{tikzcd}
\textsc{Bic} \arrow[ddr, "\text{forget } c"'] & \textsc{HG} \arrow[l, bend left, "R"] & \underline{\textsc{HG}} \arrow[l, hook'] \arrow[loop above, "duality"] \arrow[r, bend right, "R"'] & \underline{\textsc{Bic}} \arrow[l, bend right, "R'"'] \arrow[ddl, "\text{forget } c"] \\
 & \textsc{Graph} \arrow[u, hook] & \underline{\textsc{Graph}} \arrow[l, hook'] \arrow[u, hook] & \\
 & \textsc{Bip} \arrow[u, hook] & \underline{\textsc{Bip}} \arrow[l, hook'] \arrow[u, hook] &
\end{tikzcd}
$$

Equivalence of categories