A Categorical Approach to Knowledge Management

Computational Category Theory Workshop

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BAKER MOUNTAIN
Science Technology Service

Personal Background and Perspective

- Employment
  - Shepherd University. 2011–present
  - Baker Mountain Research Corporation. 2011–present
  - Metron, Inc. 2004–2011
  - University of Dallas. 2001–2004
- Education
  - Ph.D. Mathematics, University of Illinois (category theory and dynamic systems)
  - M.S. Aeronautical Engineering, University of Illinois
  - B.S. Mathematics, Aeronautical Engineering. Rensselaer Polytechnic Institute
- Project Experience
  - Consultant: Senior Hadoop Analyst for PNC Financial Services. 2015
  - Consultant: Statistical analysis and model development for Flexible Plan Investments, Bloomfield Hills, MI. 2014–present
  - Established and manages Shepherd Laboratory for Big Data Analytics
  - Co-Investigator with S. Bringsjord (RPI) and J. Hummel (UIUC): *Great Computational Intelligence*. AFOSR. 2011–14
  - PI with N. Yanofsky (CUNY): *Quantum Kan Extensions*. IARPA. 2011–12
  - Analyst. *Wide Aperture Passive Sonar Algorithm Development*. ONR. 2010
  - Technical Lead. *Exposing and Influencing Enemy Networks*. ONR. 2009–10
  - PI: *Robust Decision Making*. AFOSR. 2008–2010
  - Analyst: *TradeNet Integration into Global Trader*. ONI. 2009
  - PI: *Reasoning Under Uncertainty*. Phase I–II SBIR. MDA. 2005–8
  - PI: *Measures of Effectiveness Sensitivity Calculator*. ONR. 2006–7

Opportunities for Computational Category Theory

- Sketch theory as a complement to other semantic technologies
  - Modularity and compositionality
  - Knowledge alignment, theory of a sketch, and Carmody-Walters
  - Q-Trees as a reasoning engine
  - Automated inference of and use of context
  - Transformations between sketches, logical theories, and ontologies
  - Limitations of other technologies
- Quantum computing
  - No improvement in exact calculation of products, equalizers, pullbacks or coproducts in Set_f
  - Exact calculation of coequalizers
  - Fast, approximate algorithms
  - Quantum algorithms in other categories
- Uncertainty models
  - Assigning convex sets to logical formulae (i.e., the Eilenberg-Moore category of the Lawvere-Čencov stochastic category)
  - Categories for other uncertainty models. Corresponding logics.
  - Transformations between uncertainty models

Sketches: Historical Timeline

- 1943: Eilenberg and Mac Lane introduce category theory
- 1958: Kan introduces the concept of adjoints
- 1963: Lawvere characterizes quantifiers and other logical operations as adjoints
- 1968: C. Ehresman introduces sketch theory
- 1985: KL-ONE — First implementation of a description logic system
- 1985: Barr and Wells publish *Toposes, Triples and Theories*
- 1989: J. W. Gray publishes Category of Sketches as a Model for Algebraic Semantics
- 1990: Barr and Wells publish *Categories for Computing Science*
- 1995: Carmody and Walters publish algorithm for computing left Kan extensions
- 1999: RDF becomes a W3C recommendation
- 2000: Johnson and Rosebrugh apply sketch data model to database interoperability
- 2000: DARPA begins development of DAML
- 2001: Dampney, Johnson and Rosebrugh apply sketches to view update problem
- 2001: W3C forms the Web-Ontology Working Group
- 2004: RDFS and OWL become W3C recommendations
- 2008: Johnson and Rosebrugh release Easik software
- 2009: OWL2 becomes a W3C recommendation
- 2012: Johnson, Rosebrugh and Wood use sketches to formulate lens concept of view updates

Knowledge Technologies

Facets of Knowledge Models

Storage	Queries
Constraints	Uncertainty
Alignment	Dynamics
Context/Views	Software
Reasoning	Decision-Making
Translations	Human Interface

Knowledge Technologies

- Mathematical Logic (1879)
- Databases + SQL (1968)
- Semantic Web OWL/RDF
  + Description Logic (1999)
- Sketches (1968/2000)
  + Q-Trees (1990)

Sketch Theory: Overview

- Mature, graph-based foundation
- Vertices = classes or relations
- Edges = type information or maps
- Constraints/meta-data specified via graph maps (cones/cocones)
- Sketch maps respect constraints
- Grew from category theory in 1968
- Applied to data modeling since 1989

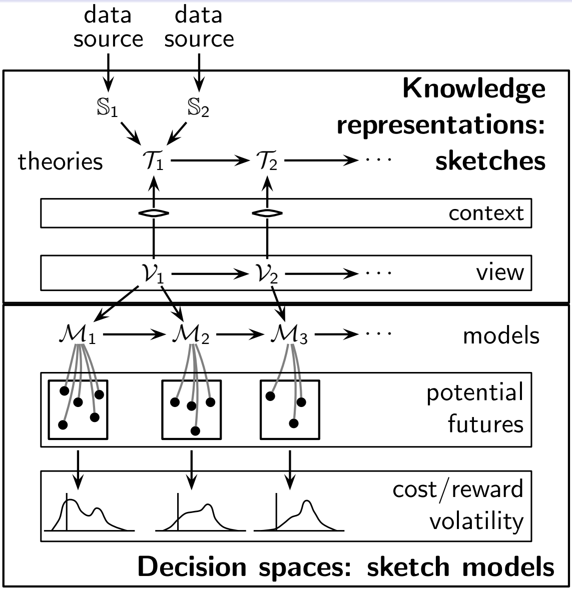
Sketch Theory: Strengths

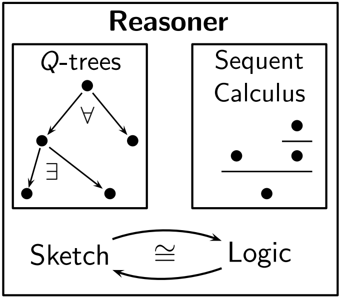
- Visual/graphical modeling
- Modularity: data/concepts/uncertainty
- Combinatory algebra of sketches
- Concise graphical inference and inter-convertibility with 1st order logic
- Derived concepts via CW algorithm
- Rich composable views/context
- Dynamics via sketch maps

Opportunity: Limitations of Knowledge Technologies

- Mathematical Logic
  - Computational complexity of some predicate calculus fragments (e.g., classical logic)
  - Complexity of the syntactic category used for knowledge alignment
  - Challenging to develop a human interface
- Databases + SQL
  - Limited notion of context/view (a single table)
  - Static schema
- Semantic Web OWL/RDF + Description Logic
  - Lack of modularity: meta-data, instance data and uncertainty integrated into a monolithic ontology
  - Limited compositional algebra: (disjoint) unions of ontologies
  - Need for constraint-preserving maps
- Sketch Theory
  - Meager computational infrastructure (e.g., relative to Jena)

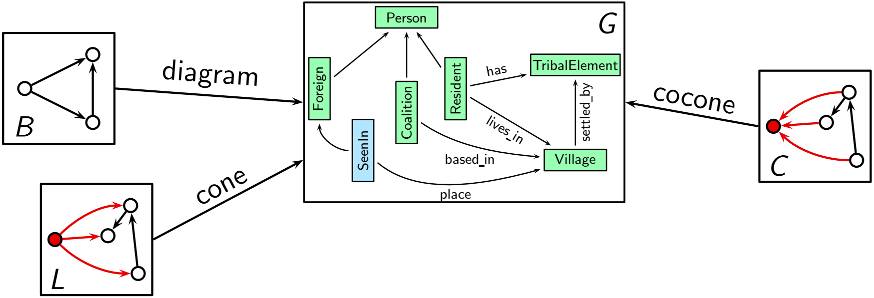
Sketch Conops





Sketch $(G, \mathcal{D}, \mathcal{L}, \mathcal{C})$

- All semantic constraints in a sketch are expressed using graph maps.
- A sketch $(G, \mathcal{D}, \mathcal{L}, \mathcal{C})$ consists of:
  - An underlying graph G
  - A set $\mathcal{D}$ of diagrams $B \to G$
  - A set $\mathcal{L}$ of cones $L \to G$
  - A set $\mathcal{C}$ of cocones $C \to G$



Categorical Semantics of Sketches

- Vertices are interpreted as objects
- Edges are interpreted as morphisms
- Classes of constraints (cones and cocones) are distinguished by the shapes of their base graphs.
- Classes of sketches are distinguished by their classes of constraints.
- Like logics and OWL species, these have different expressive powers.

Small sample of the sketch semantics landscape

Sketch Class	Set	Partial Func.	Stoch. Matrices	Čencov Cat.	Prob. 0 Refl.	Dempster Shafer	Fuzzy Sets	Convex Sets
linear	•	•	•	•	•	•	•	•
Finite Limit	•	•	×	×	×	×	•	•
Finite Coproduct	•	•	•	•	•	•	•	•
Entity-Attribute	•	•	×	×	×	×	•	•
Mixed	•	•	×	×	×	×	•	•

Sketch Maps and Model Maps

- A sketch map $\mathbb{S}_1 \to \mathbb{S}_2$ is a graph map
$$G_1 \longrightarrow G_2$$
that preserves all the constraints of $\mathbb{S}_1$.
$$B \longrightarrow G_1 \longrightarrow G_2$$
- We use sketch maps to formulate the Alignment Problem.
- Given models M_1 and M_2 of a sketch $\mathbb{S}$, a model map $M_1 \to M_2$ is a collection of functions (one for each vertex V of G)
$$M_1(V) \xrightarrow{\tau_V} M_2(V)$$
that are consistent with the edges of G.
- Example:

$$\begin{array}{c} \text{Resident} \\ \Big\downarrow \text{live\_in} \\ \text{Village} \end{array} \qquad \begin{tikzcd} M_1(\text{Resident}) \arrow[r, "\tau"] \arrow[d, "M_1(\text{lives\_in})"'] & M_2(\text{Resident}) \arrow[d, "M_2(\text{lives\_in})"] \\ M_1(\text{Village}) \arrow[r, "\tau"'] & M_2(\text{Village}) \end{tikzcd}$$

- J.W. Gray. The category of sketches as a model for algebraic semantics. 1989.

Sketch Theory: Questions

- EA sketch instance data (models) can be implemented using relational database features such as foreign keys and triggers.
- What technologies support management of distributed models of sketches?
    - Google Megastore, Tenzing or Spanner?
    - Apache Accumulo?
    - Is a graph database more appropriate?

Presentations

- A sketch | first-order theory | ontology is a presentation of knowledge.
- Presentations generate additional knowledge needed for alignment (e.g., 'uncle $=$ brother $\circ$ parent')

Logical theory $\mathbb{T}$	syntactic category $\mathcal{C}_{\mathbb{T}}$
Ontology	rules
Sketch $\mathbb{S}$	theory of a sketch $\mathbb{T}(\mathbb{S})$

- Different presentations may generate 'equivalent' structures.
- Sketches $\mathbb{S}_1$ and $\mathbb{S}_2$ representing common concepts are aligned by finding a sketch $\mathcal{V}$ and sketch maps as shown.

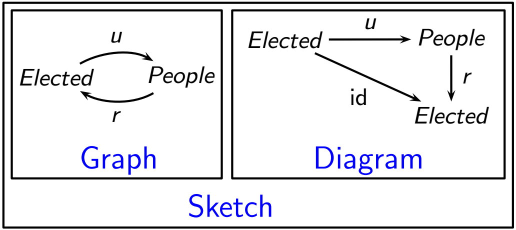
$$\begin{tikzcd}
\mathbb{S}_1 \arrow[d] & & \mathcal{V} \arrow[dll] \arrow[drr] & & \mathbb{S}_2 \arrow[d] \\
\mathcal{T}_1 \arrow[drr] & & & & \mathcal{T}_2 \arrow[dll] \\
& & \mathcal{T} & &
\end{tikzcd}$$

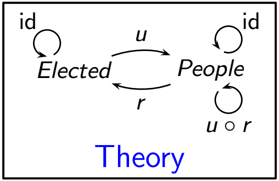
- Theory of a (linear) sketch
  - Carmody-Walters algorithm for computing left Kan extensions: generalizes Todd-Coxeter procedure used in computational group theory
  - Complexity difficult to characterize: can depend on order of constraints

Civics Sketch $\mathbb{S}_1$

First formulation of civics concepts:

- Two classes: People and Elected officials
- People have Elected representatives via r.
- Elected officials are instances of people via u.
- Elected officials represent themselves via a diagram.





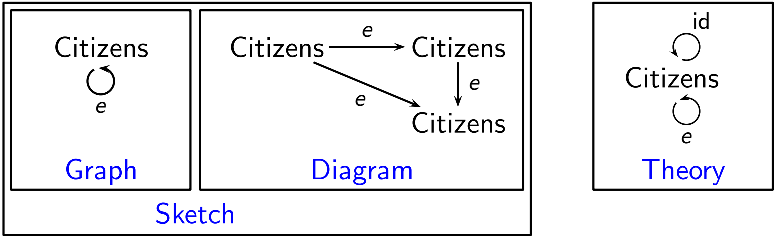
- The diagram truncates the infinite list of composites (property chains).

$$u \circ r \qquad r \circ u \qquad u \circ r \circ u \qquad r \circ u \circ r \qquad \cdots$$

Civics Sketch $\mathbb{S}_2$

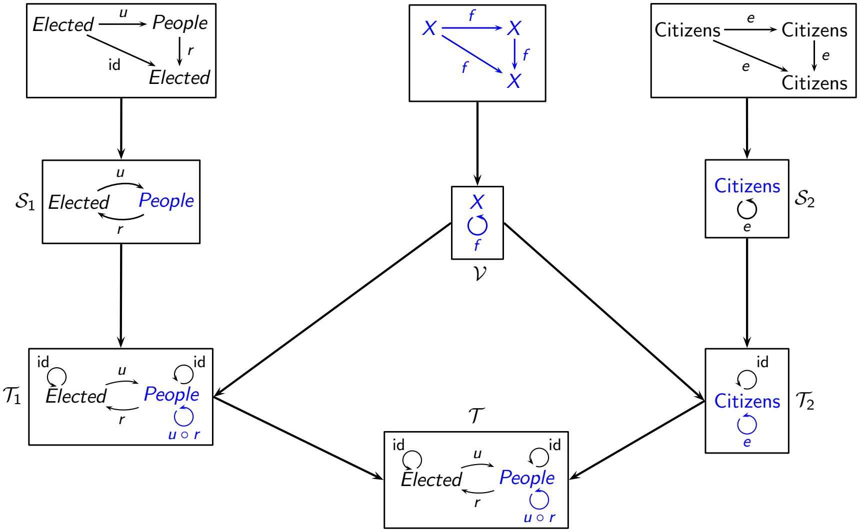
Alternative formulation of the concepts:

- One class: Citizens
- Citizens have elected representatives via *e*.
- Elected officials represent themselves via a diagram.



- Number and names of vertices in $\mathcal{S}_1$ and $\mathcal{S}_2$ differ.
- The edges u and r of $\mathcal{S}_1$ have no corresponding edges in $\mathcal{S}_2$.
- The edge e of $\mathcal{S}_2$ has no corresponding edge in $\mathcal{S}_1$.

Alignment of the Civics Sketches



Sketch Alignment: Questions

- What algorithms are available for computing the theory of a sketch?
  - Carmody-Walters for linear sketches
  - Others?
- To what extent can the sketch alignment problem be automated?
  - Find the appropriate intersection
  - Renaming of vertices and edges
- Can instance data be used to support sketch alignment?

First-Order Civics Theories $\mathbb{T}_1$ and $\mathbb{T}_2$

- $\mathbb{T}_1$
  - **Sorts**: People, Elected
  - **Function symbols**:

$$u : \text{Elected} \longrightarrow \text{People} \qquad r : \text{People} \longrightarrow \text{Elected}$$

  - **Axiom**: elected officials represent themselves

$$\top \vdash_x (r(u(x)) = x)$$

- $\mathbb{T}_2$
  - **Sorts**: Citizens
  - **Function symbols**:

$$e : \text{Citizens} \longrightarrow \text{Citizens}$$

  - **Axiom**: elected officials represent themselves

$$\top \vdash_x (e(e(x)) = e(x))$$

First-Order Logic: Sequent Calculus

Structural Rules[1]

$$(\varphi \vdash_{\vec{x}} \varphi) \qquad \frac{(\varphi \vdash_{\vec{x}} \psi)}{(\varphi[\vec{s}/\vec{x}] \vdash_{\vec{y}} \psi[\vec{s}/\vec{x}])} \qquad \frac{(\varphi \vdash_{\vec{x}} \psi)\ \ (\psi \vdash_{\vec{x}} \chi)}{(\varphi \vdash_{\vec{x}} \chi)}$$

Implication

$$\frac{((\varphi \wedge \psi) \vdash_{\vec{x}} \chi)}{(\varphi \vdash_{\vec{x}} (\psi \Rightarrow \chi))}$$

Equality

$$(\top \vdash_{x} (x = x))$$

$$((\vec{x} = \vec{y}) \wedge \varphi \vdash_{\vec{z}} \varphi[\vec{y}/\vec{x}])$$

Quantification[2]

$$\frac{(\varphi \vdash_{\vec{x},y} \psi)}{((\exists y)\varphi \vdash_{\vec{x}} \psi)} \qquad \frac{(\varphi \vdash_{\vec{x},y} \psi)}{(\varphi \vdash_{\vec{x}} (\forall y)\psi)}$$

Conjunction

$$(\varphi \vdash_{\vec{x}} \top) \qquad ((\varphi \wedge \psi) \vdash_{\vec{x}} \varphi) \qquad ((\varphi \wedge \psi) \vdash_{\vec{x}} \psi) \qquad \frac{(\varphi \vdash_{\vec{x}} \psi)\ \ (\varphi \vdash_{\vec{x}} \chi)}{(\varphi \vdash_{\vec{x}} (\psi \wedge \chi))}$$

Disjunction

$$(\bot \vdash_{\vec{x}} \varphi) \qquad (\varphi \vdash_{\vec{x}} (\varphi \vee \psi)) \qquad (\psi \vdash_{\vec{x}} (\varphi \vee \psi)) \qquad \frac{(\varphi \vdash_{\vec{x}} \chi)\ \ (\psi \vdash_{\vec{x}} \chi)}{((\varphi \vee \psi) \vdash_{\vec{x}} \chi)}$$

Distributive Law[3]

$$((\varphi \wedge (\psi \vee \chi) \vdash_{\vec{x}} (\varphi \wedge \psi) \vee (\varphi \wedge \chi))$$

Frobenius Axiom[3]

$$((\varphi \wedge ((\exists y)\psi) \vdash_{\vec{x}} (\exists y)(\varphi \wedge \psi))$$

Excluded Middle

$$(\top \vdash_{x} (\varphi \vee \neg\varphi))$$

Contexts are suitable for the formulae that occur on both sides of $\vdash$.

1 In the substitution rule, $\vec{y}$ contains all the variables of $\vec{x}$.

2 Bound variables do not also occur free in any sequent.

3 The Distributive Law and Frobenius Axiom are derivable in full, first-order logic.

Soundness

- **Soundness Theorem**: Let $\mathbb{T}$ be a Horn theory and let M be a model of $\mathbb{T}$ in a cartesian category. If $\varphi \vdash_{\vec{x}} \psi$ is provable from $\mathbb{T}$ in Horn logic, then the sequent is satisfied in M.

  Proof: Induction on inference rules using the category properties used to define semantics of terms- and formulae-in-context.
- We can replace Horn and cartesian by any combination of:

Logic	**Category**
Regular	Regular
Coherent	Coherent
First-order	Heyting
Classical first-order	Boolean coherent
Linear	$*$-autonomous
Intuitionistic higher-order	Topos
S4 modal (predicate)	sheaves on a topological space

Completeness

- **Completeness Theorem**: Let $\mathbb{T}$ be a regular theory. If $\varphi \vdash_{\vec{x}} \psi$ is a regular sequent that is satisfied in all models of $\mathbb{T}$ in regular categories $\mathcal{D}$, then it is provable from $\mathbb{T}$ in regular logic.

  Proof: Construct the syntactic category $\mathcal{C}_{\mathbb{T}}$ with a generic model $M_{\mathbb{T}}$

category of models of $\mathbb{T}$ in $\mathcal{D}$	$\cong$	category of regular functors $\mathcal{C}_{\mathbb{T}} \to \mathcal{D}$
$\mathrm{Mod}_{\mathbb{T}}(\mathcal{D})$	$\cong$	$\mathrm{Reg}(\mathcal{C}_{\mathbb{T}}, \mathcal{D})$

- We can replace regular theories and categories by any of:

Logic	**Category**
Cartesian	Cartesian
Coherent	Coherent
First-order	Heyting

- The Completeness Theorem also holds if we replace $\mathcal{D}$ by Set.

Alignment of Logical Theories

- Provable equivalence: applicable to theories over the same signature
- Theories $\mathbb{T}_1$ and $\mathbb{T}_2$ are Morita equivalent if their categories of models $\mathrm{Mod}_{\mathbb{T}}(\mathcal{D})$ (in any category $\mathcal{D}$ of the appropriate class) are equivalent.

$$\mathrm{Mod}_{\mathbb{T}_1}(\mathcal{D}) \cong \mathrm{Mod}_{\mathbb{T}_2}(\mathcal{D})$$

- Theories are Morita equivalent iff their syntactic categories are.

$$\mathcal{C}_{\mathbb{T}_1} \cong \mathcal{C}_{\mathbb{T}_2}$$

- This solves the alignment problem for the civics theories.
- It can be difficult to use in practice.
  - Types are interpreted as equivalence classes of formulae
  - Functions and relations are interpreted as equivalence classes of formuae
  - Syntactic categories are typically infinite, even for simple theories
  - No general algorithm exists
  - Could one develop a lazy algorithm?

Syntactic Categories

- Let $\mathbb{T}$ be a regular theory. There is a regular category $\mathcal{C}_{\mathbb{T}}$ that has a model of $\mathbb{T}$.

objects:	α-equivalence classes of formulae-in-context: $\{\vec{x}.\varphi\}$ where φ is regular over $\mathbb{T}$
morphisms :	$\{\vec{x}.\varphi\} \xrightarrow{[\theta]} \{\vec{y}.\psi\}$ $\theta \vdash_{\vec{x},\vec{y}} \varphi \wedge \psi \qquad \varphi \vdash_{\vec{x}} (\exists \vec{y})\theta \qquad \theta \wedge \theta[\vec{z}/\vec{y}] \vdash_{\vec{x},\vec{y},\vec{z}} (\vec{z} = \vec{y})$
composition:	$\begin{array}{ccc} \{\vec{x}.\varphi\} & \xrightarrow{[\theta]} & \{\vec{y}.\psi\} \\ & \searrow^{[(\exists\vec{y})(\theta\wedge\gamma)]} & \downarrow{[\gamma]} \\ & & \{\vec{z}.\chi\} \end{array}$
identity:	$\{\vec{x}.\varphi\} \xrightarrow{[\varphi\wedge(\vec{x}'=\vec{x})]} \{\vec{x}'.\varphi[\vec{x}'/\vec{x}]\}$

Syntactic Categories (Continued)

- $\mathcal{C}_{\mathbb{T}}$ contains a model of $\mathbb{T}$.

sorts	A	$\{x.\top\}$ for $x : A$
types	1	$\{[].\top\}$
	$A_1 \times \cdots \times A_n$	$\{\vec{x}.\top\}$ for $x_i : A_i$
function symbols	$f : A_1 \times \cdots \times A_n \to B$	$\{\vec{x}.\top\} \xrightarrow{[f(x_1,\ldots,x_n)=y]} \{y.\top\}$ for $x_i : A_i$ and $y : B$
relation symbols	$R \rightarrowtail A_1 \times \cdots \times A_n$	$\{\vec{x}.R(\vec{x})\} \longrightarrow \{\vec{x}.\top\}$

- The axioms of $\mathbb{T}$ are satisfied in this model.

Proof of $(u(x) = u(y)) \vdash_{x,y} (x = y)$ for Civics Theory $\mathbb{T}_1$

1. $(u(x) = u(y)) \vdash_{x,y} (u(x) = u(y))$ .... Id
2. $(u(x) = u(y)) \vdash_{x,y} \top$ .... $\top$
3. $\top \vdash_{x} (r(u(x)) = x)$ .... axiom
4. $\top \vdash_{x,y} (r(u(x)) = x)$ .... Sub (3)
5. $\top \vdash_{x,y} (r(u(y)) = y)$ .... Sub (3)
6. $(x = y) \wedge (r(x) = z) \vdash_{x,y,z} (r(y) = z)$ Eq1
7. $(u(x) = u(y)) \wedge (r(u(x)) = x) \vdash_{x,y,z} (r(u(y)) = x)$ .... Subs (6)
8. $(u(x) = u(y)) \wedge (r(u(x)) = x) \vdash_{x,y} (r(u(y)) = x)$ .... Subs (7)
9. $(x = y) \vdash_{x,y} (y = x)$ .... previous proof
10. $(r(u(y)) = x) \vdash_{x,y} (x = r(u(y)))$ .... Subs (9)
11. $(u(x) = u(y)) \wedge (r(u(x)) = x) \vdash_{x,y} (x = r(u(y)))$ .... Cut (8), (10)
12. $(x = y) \wedge (y = z) \vdash_{x,y,z} (x = z)$ .... previous proof
13. $(x = r(u(y))) \wedge (r(u(y)) = y) \vdash_{x,y,z} (x = y)$ .... Subs (12)
14. $(x = r(u(y))) \wedge (r(u(y)) = y) \vdash_{x,y} (x = y)$ .... Subs (13)
15. $(u(x) = u(y)) \vdash_{x,y} (r(u(x)) = x)$ .... Cut (2), (4)
16. $(u(x) = u(y)) \vdash_{x,y} (u(x) = u(y)) \wedge (r(u(x)) = x)$ .... $\wedge$I (1), (15)
17. $(u(x) = u(y)) \vdash_{x,y} (x = (r(u(y)))$ .... Cut (16), (11)
18. $(u(x) = u(y)) \vdash_{x,y} (r(u(y)) = y)$ .... Cut (2), (5)
19. $(u(x) = u(y)) \vdash_{x,y} (x = r(u(y))) \wedge (r(u(y)) = y)$ .... $\wedge$I (17), (18)
20. $(u(x) = u(y)) \vdash_{x,y} (x = y)$ .... Cut (19), (14)

Prover9 Proof

- Input file:

```
formulas(assumptions).
  all x (r(u(x)) = x).
end_of_list.
formulas(goals).
  all x all y (u(x) = u(y)) -> (x = y).
end_of_list.
```

- Proof:

```
1 (all x r(u(x)) = x) ....................# label(non_clause).  [assumption].
2 (all x all y u(x) = u(y)) -> x = y ................# label(non_clause)
                                                     # label(goal).  [goal].
3 r(u(x)) = x.  ..............................................[clausify(1)].
4 u(x) = u(y).  ...................................................[deny(2)].
5 c2 != c1.  ......................................................[deny(2)].
6 x = y.  .........................[para(4(a,1),3(a,1,1)),rewrite([3(2)])].
7 $F.  ...................................................[resolve(6,a,5,a)].
```

- The shorter proof by contradiction uses classical first-order logic.
- First-order horn logic has lower computational complexity in general.

Q-Sequences and Q-Trees (Freyd-Scedrov 1990)

- P. Freyd and A. Scedrov. *Categories, Allegories*. 1990
- A Q-sequence $\mathcal{Q} = (A, a, Q)$ in a category $\mathcal{D}$ consists of lists of
  - objects $A_0, \ldots, A_n$
  - morphisms $a_i : A_i \to A_{i+1}$ for $0 \leq i < n$
  - quantifiers $Q_0, \ldots, Q_n$

$$\boxed{A_0 \overset{Q_0}{\Big|} \quad A_1 \overset{Q_1}{\Big|} \quad \cdots \quad \overset{Q_{n-1}}{\Big|} \quad A_n \overset{Q_n}{\Big|}}$$

- $\sigma\,\mathcal{Q}$ is: $\boxed{A_1 \overset{Q_1}{\Big|} \quad \cdots \quad \overset{Q_{n-1}}{\Big|} \quad A_n \overset{Q_n}{\Big|}}$
- A morphism $A_0 \xrightarrow{f_0} B$ satisfies $\mathcal{Q}$ if one of the following holds:
  - $n = 0$ and $Q_0 = \forall$
  - $n > 0$, $Q_0 = \forall$, and for every commutative triangle
$$\begin{tikzcd} A_0 \arrow[r, "a_0"] \arrow[dr, "f_0"'] & A_1 \arrow[d, dotted, "f_1"] \\ & B \end{tikzcd}$$
    , the morphism $A_1 \xrightarrow{f_1} B$ satisfies the Q-sequence $\sigma\,\mathcal{Q}$
  - $n > 0$, $Q_0 = \exists$, and there exists a commutative triangle
$$\begin{tikzcd} A_0 \arrow[r, "a_0"] \arrow[dr, "f_0"'] & A_1 \arrow[d, dotted, "f_1"] \\ & B \end{tikzcd}$$
    for which $A_1 \xrightarrow{f_1} B$ satisfies the Q-sequence $\sigma\,\mathcal{Q}$
- Q-trees generalize Q-sequences by allowing branching.

Sketch Inference

In civics sketch $\mathbb{S}_1$, we may conclude that Elected is a subclass of People.

$$\begin{tikzcd} \mathit{Elected} \arrow[r, bend left, "u"] & \mathit{People} \arrow[l, bend left, "r"] \end{tikzcd}$$

Graph

$$\begin{tikzcd} \mathit{Elected} \arrow[r, "u"] \arrow[dr, "\mathrm{id}"'] & \mathit{People} \arrow[d, "r"] \\ & \mathit{Elected} \end{tikzcd}$$

Diagram

$$\begin{tikzcd} Z \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "+"] & \text{Elected} \end{tikzcd}$$

$\forall$

$$\begin{tikzcd} Z \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "+"] & \text{Elected} \arrow[r, "u"] & \text{People} \end{tikzcd}$$

$\exists$

$$\begin{tikzcd} Z \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "+"] & \text{Elected} \arrow[r, "u"] & \text{People} \arrow[r, "r"] & \text{Elected} \end{tikzcd}$$

$\forall$

$$\begin{tikzcd} Z \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] \arrow[r, phantom, "+"] & \text{Elected} \arrow[r, "u"] \arrow[rr, bend right, "\mathrm{id}"'] & \text{People} \arrow[r, "r"] & \text{Elected} \end{tikzcd}$$

$\exists$

$$\begin{tikzcd} Z \arrow[r, bend left, "x"] \arrow[r, bend right, "y"'] & \text{Elected} \arrow[r, "u"] \arrow[rr, bend right, "\mathrm{id}"'] & \text{People} \arrow[r, "r"] & \text{Elected} \end{tikzcd}$$

$\forall$

Sketch Reasoning: Questions

- Categories, Allegories: 1.3(10)1. *Any elementary predicate in category theory is given by a finitely presented Q-tree with a free category as root.*
- What algorithms have been developed for Q-tree inference?
- Is there a correspondence between classes of logics (Horn, regular, etc.) and classes of Q-trees?
- Is there a correspondence between classes of sketches (linear, finite limit, etc.) and classes of Q-trees?
- τ categories as a guide to implementing cartesian categories?

Transforming Sketches into First-Order Theories

- Sketches are related to first-order logical theories by theorems of the form: Given any sketch $\mathbb{S}$ of class X, there is a logical theory $\mathbb{T}$ of class Y for which $\mathbb{S}$ and $\mathbb{T}$ have equivalent classes of models.
- D2.2 of Johnstone's *Sketches of an Elephant: A Topos Theory Compendium* gives explicit constructions of $\mathbb{T}$ from $\mathbb{S}$ and conversely.

Class of Sketches	Fragment of Predicate Calculus	Logical Connectives
finite limit	cartesian	$=, \top, \wedge, \exists^*$
regular	regular	$=, \top, \wedge, \exists$
coherent	coherent	$=, \top, \wedge, \exists, \perp, \vee$
geometric	geometric	$=, \top, \wedge, \exists, \perp, \bigvee$
σ-coherent	σ-coherent	$=, \top, \wedge, \exists, \perp, \bigvee_{i=1}^{\infty}$
finitary	σ-coherent	

* In cartesian logic, only certain existentially quantified formulae are allowed.

Example: Transforming the Civics Sketches to Theories

- General construction (D2.2 of *Sketches of an Elephant* by P.T. Johnstone)
  - Vertices become sorts
  - Edges become function symbols
  - No relation symbols
  - Diagrams become axioms
  - Cones and cocones induce axiom schema
- $\mathbb{S}_1$ induces $\mathbb{T}_1$ and $\mathbb{S}_2$ induces $\mathbb{T}_2$
- Add a finite limit constraint to $\mathbb{S}_1$

$$\begin{tikzcd} & 1 \arrow[d] \\ 0 \arrow[r] & 2 \end{tikzcd}$$

base

$$\xrightarrow{\varphi}$$

$$\begin{tikzcd} \text{Elected} \arrow[r, "\mathrm{id}"] \arrow[d, "\mathrm{id}"'] \arrow[dr, "u"] & \text{Elected} \arrow[d, "u"] \\ \text{Elected} \arrow[r, "u"'] & \text{Person} \end{tikzcd}$$

All induced sequents are derivable in $\mathbb{T}_1$

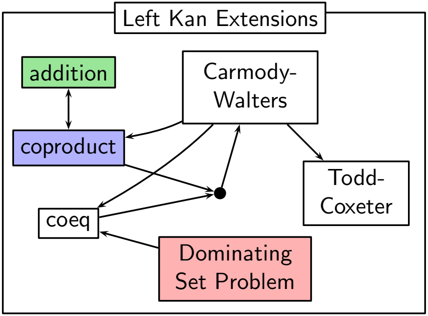
$$\top \vdash_x \big(u(x) = u(x)\big)$$
$$\top \vdash_x \big(u(x) = u(x)\big)$$
$$\big((x = y) \wedge (u(x) = u(y)) \wedge (x = y)\big) \vdash_{x,y} (x = y)$$
$$\big((u(x) = y) \wedge (u(x') = y)\big) \vdash_{x,x',y} \exists x_0\big((x_0 = x) \wedge (u(x_0) = y) \wedge (x_0 = x')\big)$$

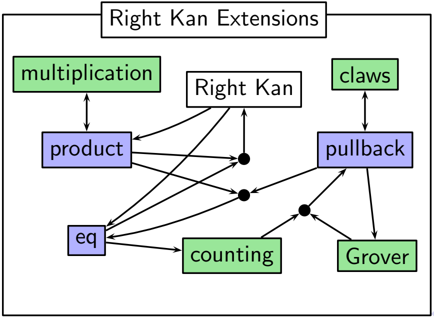
Sketch Translations: Questions

- The proof in 2.2.1 of Johnstone's *Sketches of an Elephant* of the existence of a Morita equivalent sketch for a logical theory (both of suitable classes) is not a direct construction.
- Is there an explicit (finite) construction?
- What classes of sketches correspond to OWL dialects?
- How could such mappings be used to solve the ontology alignment problem?
  - transform ontologies to sketches + instance data
  - align the sketches
  - transform back to ontologies (if necessary)

Quantum Kan Extensions

- Quantum Kan Extensions. IARPA seedling with N. Yanofsky (CUNY)
- What tasks will quantum computers be able to perform better than classical machines?
- Research findings (for Kan extensions in Set_f)
  - Known quantum algorithms
  - Exact quantum algorithms do not improve upon classical complexity
  - NP Complete problem
  - Focus on approximate algorithms, coequalizers and Kan extensions in categories with additional algebraic structure.





Kan Extensions: Definitions

- Given:
  - Categories $\mathcal{A}$ and $\mathcal{B}$ (presented as directed graphs with commutativity constraints)
  - A functor $F : \mathcal{A} \to \mathcal{B}$ (assigning a $\mathcal{B}$-path to each $\mathcal{A}$-edge)
  - An action $X : \mathcal{A} \to \mathbf{Sets}$ (assigning a set to each $\mathcal{A}$-vertex and a function to each $\mathcal{A}$-edge)
- A left Kan extension of X along F consists of:
  - An action $L : \mathcal{B} \to \mathbf{Sets}$
  - A natural transformation $\epsilon_A : X(A) \to L(F(A))$

$$\begin{tikzcd} \mathcal{A} \arrow[rr, "F"] \arrow[dr, "X"'] & \arrow[d, phantom, "\overset{\epsilon}{\nearrow}"] & \mathcal{B} \arrow[dl, dotted, "L"] \\ & \mathbf{Sets} & \end{tikzcd}$$

- These ingredients satisfy a *universal mapping property*.
- A right Kan extension has ϵ going the other way $\epsilon_A : L(F(A)) \to X(A)$.
- Mac Lane: "All concepts are Kan extensions."

Kan Extensions: Examples

Right Kan Extensions	Left Kan Extensions
products	coproducts
equalizers	coequalizers
fixed points	orbits
greatest lower bound	least upper bound
intersection	union
conjunction $\wedge$	implication $\Rightarrow$
existential quantification $\exists$	universal quantification $\forall$
left adjoints	right adjoints
limits	colimits
ends	coends
claws	coset enumeration

- Right Kan extensions can be calculated from products and equalizers.
- Left Kan extensions can be calculated from coproducts and coequalizers.

Quantum Algorithms for Pullbacks

- Classical complexity: $O(N \log X)$ ($N = \max\{X, Y\}$) to find **all** claws
  - $O(X \log X)$ comparisons to sort the values $f(x)$
  - For each y, $O(\log X)$ comparisons to search for x with $f(x) = g(y)$
- Buhrman, Dürr, Heiligman, Høyer, Magniez, Santha and de Wolf. Quantum Algorithms for Element Distinctness. 2005.
  - $O\left(X^{1/2} Y^{1/4} \log X\right)$ comparisons to (with high probability) find **a** claw (if $X \leq Y \leq X^2$) and $O(Y \log X)$ if $Y > X^2$
  - Theorem: Quantum computers cannot improve upon classical complexity of exact pullback calculations.
    - Assume we have a quantum algorithm that calculates pullbacks.
    - Given $f : X \to Y$, form the pullback P of f with itself.
    - $P - X$ counts the number of *collisions* (i.e., remove the diagonal).
    - Consequently, an efficient pullback algorithm gives an an efficient algorithm for exactly counting the number of collisions.
    - This contradicts Theorem 6.1 of the reference cited above.

Quantum Algorithms for Equalizers

- Classical complexity of finding (E, e) is $O(X)$.
- Theorem: Quantum computers can not improve upon the classical complexity of exact equalizer calculations.
  - Algorithms for equalizers and products give an algorithm for pullbacks.

$$\begin{tikzcd} & & X \arrow[dr, "f"] & \\ E \arrow[r, "e"] & X \times Y \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & Z \\ & & Y \arrow[ur, "g"'] & \end{tikzcd}$$

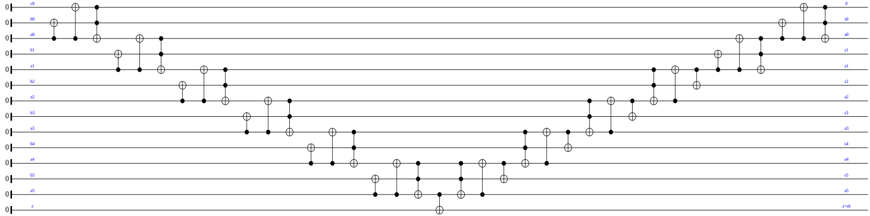
- Approximate quantum algorithm:
  - Brassard, Høyer, Mosca and Tapp. Quantum Amplitude Amplification and Estimation. 2000.
    - **Thm. 18**. `Approx_Count` with $\frac{1}{3X} < \varepsilon \leq 1$ outputs $\widetilde{E}$ with $|\widetilde{E} - E| < \varepsilon E$ with probability $2/3$ and uses an expected number of evaluations of f in the order of $\sqrt{X/\phi} + \sqrt{E(X-E)}/\phi$ where $\phi = \lfloor \varepsilon E \rfloor + 1$.
  - Childs and Eisenberg. Quantum Algorithms for Subset Finding. 2003.
    - **Thm. 1**. The query complexity of $\widetilde{E}$-subset finding is $O(X^{\widetilde{E}/(\widetilde{E}+1)})$.
  - What if `Approx_Count` miscounts?

Quantum Algorithms for Coproducts

- Adaptation of classical algorithms
  - Vedral, Barenco and Ekert. Quantum Networks for Elementary Arithmetic Operations. 1996
    - $O(n)$ depth and $O(n)$ ancillary qubits
  - Draper, Kutin, Rains and Svore. A Logarithmic-Depth Quantum Carry-Lookahead Adder. 2008
    - $O(\log(n))$ depth and $O(n)$ ancillary qubits
  - Cuccaro, Draper, Kutin and Moulton. A New Quantum Ripple-Carry Addition Circuit. 2008
    - $O(n)$ depth and 1 ancillary qubit
- Approximate Fourier transform
  - Draper. Addition on a Quantum Computer. 2000
  - Barenco, Ekert, Suominen and Törmä. Approximate Quantum Fourier Transform and Decoherence. 2008

Coproducts: Implementation (Quipper)

- Cuccaro, Draper, Kutin and Moulton. A new quantum ripple-carry addition circut. 2008. arXiv:quant-ph/0410184v1
- We can implement this in Quipper. Here is a 6-bit circuit:



```
maj :: (Qubit,Qubit,Qubit)->Circ(Qubit,Qubit,Qubit)
maj (q1, q2, q3) = do
    q2 <- qnot q2 `controlled` q3
    q1 <- qnot q1 `controlled` q3
    q3 <- qnot q3 `controlled` [q1, q2]
    return (q1, q2, q3)
```

```
uma :: (Qubit,Qubit,Qubit)->Circ(Qubit,Qubit,Qubit)
uma (q1, q2, q3) = do
    q3 <- qnot q3 `controlled` [q1,q2]
    q1 <- qnot q1 `controlled` q3
    q2 <- qnot q2 `controlled` q1
    return (q1, q2, q3)
```

- This implementation is distinct from that in the `Libraries.Arith` Quipper module which has many more ancillary qubits.
- Note: See quantum programming language references by P. Selinger

The Todd-Coxeter Coset Enumeration Algorithm

- Dehn (1911): Find an algorithm to decide whether, in a finitely-presented group, a word in the generators represents the identity element.
- Todd-Coxeter (1936): Algorithm for enumerating cosets of $H \leq G$.
  - Haselgrove (1953) gave the first computer implementation.
  - Now implemented in many computer algebra systems.
- Novikov, Boone and Britton (1955–1963): The word problem is unsolvable (in finite time by any Turing machine).
- Cannon, Dimino, Havas and Watson. Implementation and Analysis of the Todd-Coxeter Algorithm (1973).
  - Given group G and integer m, there is a presentation of G for which Todd-Coxeter will generate at least m cosets.
  - The number of cosets generated by Todd-Coxeter can vary with the order of the relations in the presentation.
- Carmody-Walters (1995): Left Kan extension algorithm for finitely-presented groups generalizes Todd-Coxeter.

Calculation of Left Kan Extensions when $\mathcal{C} = \textbf{Sets}$

$$L(B) = \left(\sum_{A \in \mathcal{A}} \mathcal{B}(F(A),\, B) \times X(A) \right) \Big/ \sim \qquad L(\beta)[b,\, x] = [\beta \circ b,\, x]$$

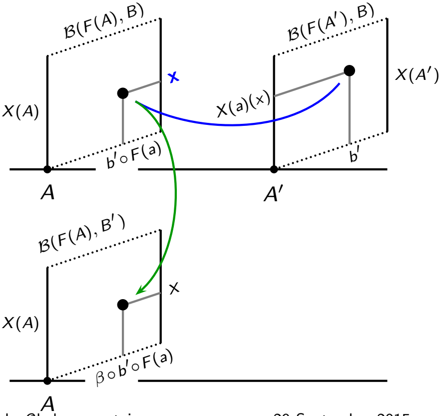
$$A \xrightarrow{a} A'$$

$$\begin{tikzcd} F(A) \arrow[r, "F(a)"] \arrow[dr, "b"'] & F(A') \arrow[d, "b'"] \\ & B \arrow[d, "\beta"] \\ & B' \end{tikzcd}$$

$$X(A) \xrightarrow{X(a)} X(A')$$

$$\begin{tikzcd} L(B) \arrow[d, "L(\beta)"'] \\ L(B') \end{tikzcd}$$

$$(b' \circ F(a),\, x) \sim (b',\, X(a)(x))$$



$$\begin{tikzcd} {[b,\, x]} \arrow[d, mapsto] \\ {[\beta \circ b,\, x]} \end{tikzcd}$$

Carmody-Walters Algorithm: Sample Calculation

Setup	Generators	Relation	Category

Setup

$$\begin{tikzcd} {|\mathcal{B}|} \arrow[r, mapsto] \arrow[dr, "X"'] & \mathcal{B} \arrow[d, dotted, "L"] \\ & \textbf{Sets} \end{tikzcd}$$

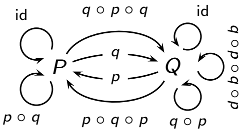
Generators

$$\begin{tikzcd} P \arrow[r, bend left, "q"] & Q \arrow[l, bend left, "p"] \end{tikzcd}$$

Relation

$$\begin{tikzcd} P \arrow[rr, "\mathrm{id}"] \arrow[d, "q"'] & & P \\ Q \arrow[r, "p"'] & P \arrow[r, "q"'] & Q \arrow[u, "p"'] \end{tikzcd}$$

Category



ϵ-tables

$X(P) \to L(P)$	
1	1

$X(Q) \to L(Q)$	
1	1

$L(P)$ = arrows into P

$L(Q)$ = arrows into Q

L-tables

$L(P) \to L(Q)$	
1	2
2	3
3	4
4	5

$L(Q) \to L(P)$	
1	2
2	3
3	4
4	5
5	6

Relation-table

$L(P) \to$	$L(Q) \to$	$L(P) \to$	$L(Q) \to$	$L(P)$	$L(P)$
1	2	3	4	5	1
2	3	4	5	6	2
3	4	1	2	3	3
4	5	2	3	4	4

Uncertainty

- An incomplete list of mathematical models of uncertainty:
  - Probabilities, fuzzy sets, rough sets, vague sets, convex sets, intervals, upper and lower probabilities, sets of probabilities, higher-order probabilities, imprecise probabilities, fuzzy measures, inner measures, outer measures, hints, boolean opinions of experts, probabilistic opinions of experts, Dempster-Shafer belief functions, Spohnian disbelief functions, plausibility functions, ranking functions, possibility functions, propositional logic, predicate logic, higher-order logic, linear logic, intuitionistic logic, modal logics, temporal logics, default logic, relative likelihoods, likelihood logic, conditional logic, Bayesian networks, credal networks, neural networks, gambles, . . .
- Program:
  - Find category-theoretic formulations
  - Derive mappings
  - Establish logical properties
- D. Scott and P. Krauss. Assigning Probabilities to Logical Formulas. 1966
- "Apparently what is needed is a new interpretation of if-then statements"
  J. Pearl. *Probabilistic Reasoning in Intelligent Systems*. 1988

Mes_Π

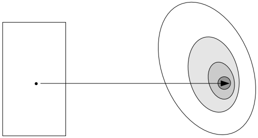
- Lawvere (1960s, unpublished), Čencov (1982), Giry (1985)
- Mes_Π = Stochastic category or category of statistical decisions
- Appears as the semantic category for the probabilistic functional programming language λ_O. Pfenning, Park, and Thrun. A Probabilistic Language Based on Sampling Functions. 2004
- (Π, η, μ) is a monad on Mes with

$$\eta_{\mathcal{X}} = \text{Dirac measure} \quad \text{and} \quad \mu_{\mathcal{X}}(\varpi)(A) = \int_X ev_A \, d\varpi$$

- Mes_Π is the Kleisli category of the monad.

$$\begin{tikzcd}
 & & \Pi^2(\mathcal{Z}) \arrow[d, "\mu_{\mathcal{Z}}"] \\
 & \Pi(\mathcal{Y}) \arrow[ur, "\Pi(g)"] & \Pi(\mathcal{Z}) \\
\mathcal{X} \arrow[ur, "f"] & \mathcal{Y} \arrow[ur, "g"'] & \mathcal{Z}
\end{tikzcd}$$

$$\begin{tikzcd}
 & \text{convex sets} \arrow[dr] & \\
\mathsf{Mes} \arrow[ur] \arrow[dr] & & \mathsf{Mes} \\
 & \mathsf{Mes}_\Pi \arrow[ur] \arrow[uu, dotted] &
\end{tikzcd}$$

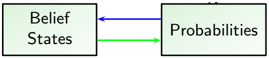


Dempster-Shafer Theory

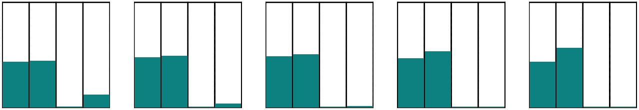
- Adapt Wendt's categogory M0RP of measure-zero reflecting maps between probability spaces.
- Define the category P0R of plausibility-zero reflecting maps between Dempster-Shapfer spaces.
- Obtain a faithful D : M0RP $\rightarrow$ P0R and a faithful V : P0R $\rightarrow$ M0RP (via the Voorbraak map) satisfying the following with V right adjoint to D.

$$
\begin{tikzcd}
 & \mathrm{P0R} \arrow[d, "V"] \\
\mathrm{M0RP} \arrow[ur, "D"] \arrow[r, "1"'] & \mathrm{M0RP}
\end{tikzcd}
$$

Belief States and Bayesian Belief States

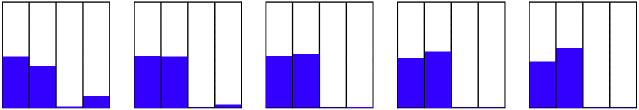


- Voorbraak map
  - Consistent with Dempster's and Bayes' Rules
  - Unnormalized version used by MIT LL and IDA



$\mathfrak{v}(m)$ $\mathfrak{v}(m \oplus m)$ $\mathfrak{v}(m \oplus m \oplus m)$ $\mathfrak{v}(\oplus_{10} m)$ $\mathfrak{v}(\oplus_{20} m)$

- Pignistic map
  - Inconsistent with Dempster's and Bayes' Rules
  - One method used by Rayetheon and Northrup-Grumman



$\mathfrak{p}(m)$ $\mathfrak{p}(m \oplus m)$ $\mathfrak{p}(m \oplus m \oplus m)$ $\mathfrak{p}(\oplus_{10} m)$ $\mathfrak{p}(\oplus_{20} m)$

Other Potential Research Directions

- Context inference and ranking
- Computational linguistics with the Lambek calculus
- Information fusion using the stochastic category

Challenge: Construct Views Tailored to Contexts

- Research area with narrower scope: context-sensitive Internet search
  - Google patent for "methods, systems and apparatus including computer program products, in which context can be used to rank search results" (USPTO 8,209,331 — 2012)
  - Yandex personalized web search challenge: www.kaggle.com
- User-selected context in IBM's Watson
- Techniques to infer context from activities and rank data elements
  - Variable-length hidden Markov model
  - Parametric models of users
  - RankNet, LambdaRank, RankSVM
- Performance metrics used for context-sensitive rankings
  - Normalized discounted cumulative gain (scoring in Kaggle competition)
  - Kendall's τ comparison of rankings
  - Jaccard distance between top N rankings and target

Papers

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