Horn clause logic for subatomics

Radhakrishnan Balu
Computational and Information Science Directorate,
US Army Research Lab, MD

09/28/2015, NIST, MD Computational category theory
Outline of the talk

- Classical probability and logic (PRISM)
- Herbrand bases and H-interpretations (LHM)
- Quantum probability spaces
- Measurable functions for grounded predicates (closed)
- Extend the result to wff with single quantifier + induction
- $H_{ES}$ space, QM observables as propositions
- Examples from quantum communication protocols
Entanglement Semantics

- Non-commutative Probability
- Quantum Probability Space
- Attach Probability Amplitudes to H-interpretations
- Projections of H
- $\rho$ - State

Denotational Semantics

Distribution Semantics

Entanglement Semantics

First-Order Probabilistic Logic

(3 = 5)  Pope(X)

X is Albert Einstein

X is Francis

X is DiNardo (probable)

X is Parolin (more or less probable)
Classical Probability

- Axiomatic theory developed by Kolmogorov
- A sample space
- $\sigma$- algebra
- Probability measure
- Product spaces, measures etc

$$CP = (\Omega, \sigma(\Omega_F), \rho)$$
Theorem (Fenstad 1967). Let $\Phi, \varphi$ be formulas in a language $L$. Suppose probabilities are assigned to formulas to satisfy the following:

i. $P(\Phi \lor \varphi) + P(\Phi \land \varphi) = P(\Phi) + P(\varphi)$

ii. $P(\neg \Phi) = 1 - P(\Phi)$

iii. $P(\Phi) = P(\varphi)$, if $\vdash \Phi \leftrightarrow \varphi$

iv. $P(\Phi) = 1$, if $\vdash \varphi$

Then there is a $\sigma$-additive probability measure $\lambda$ on the set $\Omega$ of models on the Herbrand universe $H$ for $L$. There is also $\forall \omega \in \Omega$, a probability $\mu_\omega$ on the sets of $\varphi[\omega]$ such that

$$P(\phi) = \int_S \mu_\omega(\phi[\omega]) d\lambda(\omega)$$

Probability measures on models
Probability measures on formulas

A set of probabilistic atoms $F$ and definite clauses (Horn) $R$

Probability measure on $H$-interpretations of $F$

Extend it to $DB = \{F\} \cup \{R\}$ using $^1$Least Herbrand Models

Sampling $F$ gives atoms $F'$ with TRUE assignments

$F'$ and Least Herbrand Models give TRUE atoms in $\{F\} \cup \{R\}$


$$I_j = T^j, T(I) = \{A \mid A \leftarrow B_1 \land \ldots \land B_h \ (h \geq 0)\}, \{B_1 \ldots B_h\} \subseteq I$$

Minimal model that contains all the information
The Nation’s Premier Laboratory for Land Forces

Distribution Semantics

- Enumerate members of Herbrand base
- Binary encode H-interpretations for enumeration
- Build a \( \sigma \)-algebra on the resulting Boolean lattice
- Classical Probability Space well Defined
- Attach probabilities \( P_F \) to H-interpretations of \( F \)
- Extend \( P_F \) to H-interpretations of all ground atoms
- Probability distributions with support on LHM

\[
\begin{array}{ccc}
A & B & A \land B \quad A \lor B \\
T & F & F \quad T \\
F & T & F \quad T \\
T & T & T \quad T \\
F & F & F \quad F \\
\end{array}
\]

\[
CP = \left( \bigcup_{j=1}^{\infty} I_j, \sigma(\Omega_F), \rho \right)
\]

\[
\Omega_F = \prod_{i=1}^{\infty} \{0, 1\}
\]
Quantum classical probability correspondence

- Every classical r.v. can be realized as an observable
- Classical stochastic process $\leftrightarrow$ operator process
- Non-commuting observables – main difference
- Coin Toss R.V is stochastically $\sim \sigma^x$ (Pauli operator)

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$
Hilbert space and the associated commutative von Neumann algebra (measurable functions)

\[
H_{ES} = L^2 \left( \Omega = \bigcup_{j=1}^{\infty} I_j, \sigma(\Omega_F), \rho \right)
\]

\[
\langle f, g \rangle = \int_{\Omega} (fg) d\rho, f, g \in H
\]

\[
A_{ES} = L^\infty \left( \Omega = \bigcup_{j=1}^{\infty} I_j, \sigma(\Omega_F), \rho \right)
\]
Every closed wff is measurable:
\[ \Phi^{\text{wff}}(\omega) = 1 \text{ if } \omega \vdash \Phi \]
\[ = 0 \text{ otherwise} \]

- Hilbert space \( H_{ES} \) is separable.
- Logic is the standard one.
- Points of \( H_{ES} \) are functions with support on LHM
- Members of the algebra are observables \( (\Phi^{\text{wff}}) \)
Non-Boolean lattice

\[ H = C^2; \quad e_1 = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad e_2 = |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \]

- Lattice contains 6 elements
- Maximal element 1 is projection along \( e_1 \).
- Minimal element 0 is projection along \( e_2 \).
- Projections \( p \) and \( q \) are along vectors at an angle \( \pm \Theta \) to \( e_1 \).
- Projections \( p' \) and \( q' \) are orthogonal to \( p \) and \( q \) respectively.
The Nation's Premier Laboratory for Land Forces

Non-Boolean Lattice

The Hasse diagram of the logic with incompatible observables resembling a "Chinese lantern".

We can choose angle $\Theta$ such that probabilities for $p$ and $q$ are $0.999999$. Still, their Joint probabilities is zero as $p \land q = 0$.

Radhakrishnan Balu: quantum probabilistic logic based computation and computing, e-Book to be published by SPIE (Fall 2015)
Fundamental result in Quantum probability

Every probability measure can be represented as a state using unit vector (density matrix)

Theorem 1 (Gleason): The set $S$ of probability distributions on $\mathcal{P}(H)$ is convex. If $\dim H \geq 3$ an element $\mu \in S$ is external if and only if there is a unit vector $u$ such that $\mu(P) = \langle u, Pu \rangle$ for all $P \in \mathcal{P}(H)$.

\footnote{Gleason, “Measures on the closed subspaces of Hilbert spaces.”, J. Math. Mechanics, 6 (1957)}
Proposition\textsuperscript{1}: Let $C = ((c_{ij}))$ be a positive definite matrix with $c_{ii} = 1$ for each $i, j \leq n$. Then there exist a positive integer $k \leq n$, spin observables $X_i$, $1 \leq i \leq n$ and a pure state $u$ in a Hilbert space of dimension $k$ such that:

$$\langle u, X_i u \rangle = 0, \langle u, X_i X_j u \rangle = c_{ij}, 1 \leq i, j \leq n$$

In the case of the covariance matrix with $n = 3$ the usual Bell state basis spans the Hilbert space as shown below:

$$H_1 = C^2, H_2 = C^2, H = H_1 \otimes H_2$$

$$\Phi^\pm = \frac{1}{\sqrt{2}} \left( |0\rangle_A \otimes |0\rangle_B \pm |1\rangle_A \otimes |1\rangle_B \right)$$

$$\Psi^\pm = \frac{1}{\sqrt{2}} \left( |0\rangle_A \otimes |1\rangle_B \pm |1\rangle_A \otimes |0\rangle_B \right)$$

\textsuperscript{1}K. R. Parthasarathy: An Introduction to Quantum Stochastic Calculus, Birkhauser, Basel (1992)
Extension to first order logic via induction

\[ P_{DB}(\varphi(t_1), \cdots, \varphi(t_n)) = P_{DB}(\varphi(t_{i_1}), \cdots, \varphi(t_{i_n})) \]

\[ \sum_{x_{n+1}} P_{DB}(\varphi(t_1), \cdots, \varphi(t_n) = x_{n+1}) = P_{DB}(\varphi(t_1), \cdots, \varphi(t_n)) \]

1. Tensor product of quantum states w.r.t a stabilizing sequence
2. Use Kolmogorov extension theorem then Spectral theorem

\[ \{\forall, \exists\} \oplus \{\forall, \exists\} = \{\forall, \exists, \exists, \forall\} \]
module Qubit
1. state : [0..3]; // 0 is |0>, 1 is |1>, 2 is |+>, 3 is |->
2. result : [0..1]; // Result of measurement in standard basis
3. state-1 = 2 ← (state = 0), [σ_X + σ_Z]);
4. state-1 = 3 ← (state = 1), [σ_X + σ_Z]);
5. state-1 = 0 ← (state = 2), [σ_X + σ_Z]);
6. state-1 = 1 ← (state = 3), [σ_X + σ_Z]);
7. state-1 = 0, result-1 = 0 ← (state = 0), [σ_Z]);
8. state-1 = 1, result-1 = 1 ← (state = 1), [σ_Z]);
9. state-1 = 0, result-1 = 0 ← (state = 2), [σ_Z]);
10. state-1 = 1, result-1 = 1 ← (state = 2), [σ_Z]);
endmodule

Implicit universal quantification over qudit registers
Conditional Expectation

- Commutant of $A$ is the set of bounded linear operators of $H$, members commute with every element of $A$.
  
  $$A' = \{ C : [C, A] = 0, C \in B(H) \}$$

- $A'$ need not be a commutative algebra

- Conditional expectations are defined w.r.t commutants

- Formal expression and the corresponding Horn clauses:

  $$P[D \mid A] = \sum_i \frac{P(DA_i)}{P(A_i)} A_i ; D \in A'$$

  $$P[D \mid A] \iff (D \in A') \land \sum_i \frac{P(DA_i)}{P(A_i)} A_i \land \text{spec}(A) = \{A_i\}$$

  $$P[D \mid A] \iff (D \in A'), \frac{P(DA_i)}{P(A_i)} A_i, A_i \in \text{spec}(A)$$
System space, Probe space, and composite

\[(N = C^n \otimes N_p = C^m, P(X) = Tr\{\rho X\} \otimes P_p)\]

\[A = \sum_{a \in \text{spec}(A)} aP_a; \quad (A \otimes I) \Leftrightarrow (U^*(I \otimes A')U)\]

We are copying observable A to A' in another HS

Not violating no-cloning theorem

\[P_a' = \psi_a \psi_a^*; \quad U = \sum_{a \in \text{spec}(A)} P_a \otimes X_{ap}'\]

\[X_{ab}' = \psi_b \psi_a^* + \psi_a \psi_b^* + \sum_{c \neq a,b} \psi_c \psi_c^* ; X_{aa}' = I;\]
Horn clauses for incompatibles $A, B$

$$U^* \left( I \otimes P'\right) U = P_c \otimes P'_p + (1 - P_c) \otimes P'_c \text{ if } (c \neq p).$$

$$U^* \left( I \otimes P'\right) U = \sum_a P_a \otimes P'_a.$$

$$\left( P \otimes P_p \right) \left( U^* \left( I \otimes P'\right) U \right) \left( P_c \otimes I \right) = 1, \forall c.$$
Horn clauses for incompatibles A,B

\[
U^* (I \otimes P'_c) U = P_c \otimes P'_p + (1 - P_c) \otimes P'_c
\]

probe \((B, A')\) \iff \[ U^* (I \otimes A') U, U^* (B \otimes I) U \] = 0;

\[
P[U^* (B \otimes I) U | A] \iff \text{probe}(B, A'), \sum_i \frac{P_p(BA_i)}{P_p(A_i)} U^* (I \otimes A_i) U, \text{spec}(A) = \{A_i\};
\]

Quantum state = \[
(P \otimes P_p; P_p(X) = \text{tr} \{ X A_p' \}, p' \in \text{spec}(A'))
\]

Horn clauses for incompatibles A, B, and C

\[(A \otimes I \otimes I) \iff (U^* (I \otimes A' \otimes I) U)\]

\[(B \otimes I \otimes I) \iff (U^* (I \otimes I \otimes B') U)\]

probe \((B, A') \iff [U^* (I \otimes A' \otimes I) U, U^* (B \otimes I \otimes I) U] = 0;\)

probe \((C, B') \iff [U^* (I \otimes I \otimes B') U, U^* (C \otimes I \otimes I) U] = 0;\)

Quantum state = \[\left( P \otimes P_p \otimes P_q \right)\]
Horn clauses for incompatibles $A, B,$ and $C$

Probe $A$ followed by measurement $A$ – same answer

Probe $A$, probe $B$ followed by measurement $A$
Suppose $C=A$; Different answers

\[
\begin{align*}
probe \ (B, A') & \iff \left[ U^\dagger (I \otimes A' \otimes I) U, U^\dagger (B \otimes I \otimes I) U \right] = 0; \\
probe \ (A, B') & \iff \left[ U^\dagger (I \otimes I \otimes B') U, U^\dagger (A \otimes I \otimes I) U \right] = 0;
\end{align*}
\]

Quantum state =
\[
\left( P \otimes P_p \otimes P_q \right)
\]
Summary and Future Work

- Probabilistic logic programming language for quantum h/w
- Turing computable and constructive logic
- Supports infinite probability distributions
- Extension to squeezed and non-Gaussian states
- To express and verify properties of more complex protocols
- Enriching the language with types
- Hybrid classical-quantum theorem prover
- Formalizing (Category theory based) quantum measurements as part of theorem proving process
Thank you for your attention